Antibandwidth and Cyclic Antibandwidth of Hamming Graphs

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Abstract

The antibandwidth problem is to label vertices of graph \( G(V, E) \) bijectively by integers 0, 1, ..., \(|V| - 1\) in such a way that the minimal difference of labels of adjacent vertices is maximised. In this paper we study the antibandwidth of Hamming graphs. We provide labeling algorithms and tight upper bounds for general Hamming graphs \( \Pi_{k=1}^d K_{n_k} \). We have exact values for special choices of \( n_i \)'s and equality between antibandwidth and cyclic antibandwidth values.

Keywords: antibandwidth, Hamming graph

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1 Introduction

The antibandwidth problem is to label vertices of graph $G(V,E)$ by integers $0, 1, ..., |V| - 1$ in such a way that the minimal difference of labels of adjacent vertices is maximised. This problem was originally introduced in [6] in connection with multiprocessors scheduling problems. Another motivation comes from the area of frequency assignment problems [4]. This problem is dual to the well-known bandwidth minimization problem [3] and also belongs to the large family of graph labelling problems. The antibandwidth problem is $NP$-complete. So far there exists polynomial algorithms for 3 classes of graphs: the complements of interval, arborescent comparability and threshold graphs [2,5]. Known results on antibandwidth include exact or at least asymptotically exact values for paths, cycles, complete trees [1], meshes [9,10], tori and hypercubes [8]. The cyclic antibandwidth is a natural and typical extension of the original problem when the guest graph is a cycle. The value of the cyclic antibandwidth is determined for meshes, toroidal meshes and hypercubes in [8]. Most known results are for bipartite graphs. In this paper we provide antibandwidth values for Hamming graphs. This class of graphs is interesting because of its connection to the area fo the error-correcting codes and association schemes.

Let $f$ be a one-to-one labelling

$$f : V \rightarrow \{0, 1, 2, 3, ... |V| - 1\}$$

Define the antibandwidth of $G$ according to $f$ as

$$ab(G,f) = \min_{uv \in E} |f(u) - f(v)|.$$ 

The antibandwidth of $G$ is defined as

$$ab(G) = \max_f ab(G,f).$$

Define the cyclic antibandwidth of a connected graph $G$ according to $f$ as

$$cab(G,f) = \min_{uv \in E}\{|f(u) - f(v)|, |V| - |f(u) - f(v)|\}.$$ 

The cyclic antibandwidth of $G$ is defined as

$$cab(G) = \max_f cab(G,f).$$

The $d$-dimensional Hamming graph $\Pi_{k=1}^d K_{n_k}$ is defined as the Cartesian product of $d$ complete graphs $K_{n_k}$, for $k = 1, 2, ..., d$. The vertices of $\Pi_{k=1}^d K_{n_k}$ are $d$-tuples $(i_1, i_2, ..., i_d)$, where $i_k \in \{0, 1, 2, ..., n_k - 1\}$. Two vertices $(i_1, i_2, ..., i_d)$ and $(j_1, j_2, ..., j_d)$ are adjacent iff the two tuples differ in precisely one coordinate. In case $n_k = n$, for all $k$, we denote the graph as $K_n^d$. Define the value of $N_k$ as follows. Set $N_0 = 1$ and for $k = 1, 2, ..., d$, denote $N_k = n_1 n_2 ... n_k$.

2 Hamming Graphs

**Theorem 2.1** For $d \geq 2$ and $2 \leq n_1 \leq n_2 \leq ... \leq n_d$,

$$ab(\Pi_{k=1}^d K_{n_k}) = n_1 n_2 ... n_{d-1}, \quad \text{if} \quad n_{d-1} \neq n_d, \quad d \geq 2$$

$$ab(\Pi_{k=1}^d K_{n_k}) = n_1 n_2 ... n_{d-1} - 1, \quad \text{if} \quad n_{d-1} = n_d \quad \text{and} \quad n_{d-2} \neq n_{d-1}, \quad d \geq 3$$

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and
\[ n_1n_2...n_{d-1} - \min(n_1n_2...n_{d-2}, n_{q+1}...n_{d-1}) \leq ab(\Pi_{k=1}^d K_{n_k}) \leq n_1n_2...n_{d-1} - 1, \]

where \( n_{d-2} = n_{d-1} = n_d, d \geq 3 \) and \( q \) is minimal index such that \( q \leq d - 2 \) and \( n_q = n_d \).

**Proof.** Upper bound. We use an alternative definition of the antibandwidth problem: the vertices of \( \Pi_{k=1}^d K_{n_k} \) are placed on a line bijectively into integer points \( 0, 1, 2, ..., N_d - 1 \), such that the minimum distance of adjacent vertices is maximized. Every vertex belongs to a clique \( K_{n_d} \). Consider the vertex placed at the position \( ab(\Pi_{k=1}^d K_{n_k}) - 1 \) and the corresponding clique \( K_{n_d} \). Clearly, all vertices of this clique must lie in the interval \([ab(\Pi_{k=1}^d K_{n_k}) - 1, N_d - 1]\). Observe that
\[ ab(\Pi_{k=1}^d K_{n_k}) \leq \frac{N_d - 2 - (ab(\Pi_{k=1}^d K_{n_k}) - 1)}{n_d - 1}, \]

which gives \( ab(\Pi_{k=1}^d K_{n_k}) \leq N_d - 1 \).

In case \( n_{d-1} = n_d \) we can slightly improve the argument. Every vertex belongs to two cliques: \( K_{n_d} \) and \( K_{n_{d-1}} \), whose intersection is precisely that vertex. Consider the vertex placed at the position \( ab(\Pi_{k=1}^d K_{n_k}) - 1 \) and the corresponding cliques \( K_{n_d} \) and \( K_{n_{d-1}} \). Clearly, all vertices of these cliques must lie in the interval \([ab(\Pi_{k=1}^d K_{n_k}) - 1, N_d - 1]\). Because \( n_{d-1} = n_d \), one of the cliques, say \( K_{n_{d-1}} \), must lie in a shorter interval, otherwise the two cliques would have 2 vertices in common. Hence
\[ ab(\Pi_{k=1}^d K_{n_k}) \leq \frac{N_d - 2 - (ab(\Pi_{k=1}^d K_{n_k}) - 1)}{n_d - 1} \leq N_d - 1 - \frac{1}{n_d}, \]

which implies \( ab(\Pi_{k=1}^d K_{n_k}) \leq N_d - 1 - 1 \).

**Lower bound.** Case I. Let \( n_{d-1} \neq n_d \). Define a labelling of the vertices of the Hamming graph as
\[
\begin{align*}
\quad f(i_1, i_2, ..., i_d) = ((N_d-1 + N_{d-1}/N_1)i_1 + (N_d-1 + N_{d-1}/N_2)i_2 + (N_d-1 + N_{d-1}/N_3)i_3 + \\
&+ (N_d-1 + N_{d-1}/N_{d-1})i_{d-1} + N_d-i_d) \mod N_d.
\end{align*}
\]

We show that \( f \) is a bijection. Assume that for two vertices \((i_1, i_2, ..., i_d) \neq (j_1, j_2, ..., j_d)\)
\[
\begin{align*}
\quad f(i_1, i_2, ..., i_d) = f(j_1, j_2, ..., j_d).
\end{align*}
\]

It follows that
\[
\begin{align*}
\quad (N_d-1 + N_{d-1}/N_1)i_1 + ... + (N_d-1 + 1)(i_d-1 - j_d-1) + N_d-i_d-j_d = 0 \mod N_d.
\end{align*}
\]

Observe that \( i_d-1 - j_d-1 \) is divisible by \( n_{d-1} \). As \( |i_d-1 - j_d-1| < n_{d-1} \), we have \( i_d-1 = j_d-1 \). Divide the congruence by \( n_{d-1} \). Then \( i_d-2 - j_d-2 \) is divisible by \( n_{d-2} \). As \( |i_d-2 - j_d-2| < n_2 \), we have \( i_d-2 = j_d-2 \). Divide the congruence by \( n_2 \) and continue in a similar way up to \( i_1 = j_1 \). Finally, this implies that \( i_d = j_d \), which is a contradiction. Distinguish 2 cases.
a) Consider an edge of the \( k \)-th dimension, \( k \leq d - 1 \), i.e., an edge between \((i_1, i_2, ..., i_k, ..., i_d)\) and \((i_1, i_2, ..., j_k, ..., i_d)\). Wlog assume that
\[
\begin{align*}
\quad f(i_1, i_2, ..., i_k, ..., i_d) \geq f(i_1, i_2, ..., j_k, ..., i_d).
\end{align*}
\]

Then compute
\[
\begin{align*}
\quad f(i_1, i_2, ..., i_k, ..., i_d) - f(i_1, i_2, ..., j_k, ..., i_d) = (N_d-1 + N_{d-1}/N_k)(i_k - j_k) \mod N_d.
\end{align*}
\]

If \( i_k > j_k \) then
\[
\begin{align*}
\quad (N_d-1 + N_{d-1}/N_k)(i_k - j_k) \mod N_d \geq N_d - N_{d-1}/N_k N_q > N_d-1.
\end{align*}
\]

If \( i_k < j_k \) then
\[(N_{d-1} + N_{d-1}/N_k)(i_k - j_k) \mod N_d = N_d - (N_{d-1} + N_{d-1}/N_k)(j_k - i_k) \geq N_d - (N_{d-1} + N_{d-1}/N_k)(n_k - 1) = N_{d-1}(n_d - n_k + 1 - 1/N_k + 1/N_k) > N_{d-1}.\]

b) Consider an edge of the \(d\)-th dimension. Wlog assume that 
\[f(i_1, i_2, \ldots, i_d) \geq f(i_1, i_2, \ldots, j_d).\]
Compute 
\[f(i_1, i_2, \ldots, i_d) - f(i_1, i_2, \ldots, j_d) = N_{d-1}(i_d - j_d) \mod N_d.\]
If \(i_d > j_d\) then 
\[N_{d-1}(i_d - j_d) \mod N_d = N_d - N_{d-1}(j_d - i_d) \geq N_d - N_{d-1}(n_d - 1) = N_{d-1}.\]
Case II. Let \(n_{d-1} = n_d\). Let \(q\) be the minimal index s.t. \(q \leq d - 1\) and \(n_q = n_d\). Define a labelling of the vertices of the Hamming graph as 
\[f(i_1, i_2, \ldots, i_d) = ((N_{d-1} + N_{d-1}/N_1)i_1 + \ldots + (N_{d-1} + N_{d-1}/N_{q-1})i_{q-1} + (N_{d-1} - N_{d-1}/N_q)i_q + \ldots + (N_{d-1} - N_{d-1}/N_{d-1})i_{d-1} + N_{d-1}i_d) \mod N_d.\]
Using the same method as above, we can easily show that \(f\) is a bijection. Distinguish 3 subcases.
a) Consider an edge of the \(k\)-th dimension, \(k \leq q - 1\). The analysis is the same as in Case I.a).
b) Consider an edge of the \(k\)-th dimension, \(q \leq k \leq d - 1\). Wlog assume that 
\[f(i_1, i_2, \ldots, i_k, \ldots, i_d) \geq f(i_1, i_2, \ldots, j_k, \ldots, i_d).\]
Compute 
\[f(i_1, i_2, \ldots, i_k, \ldots, i_d) - f(i_1, i_2, \ldots, j_k, \ldots, i_d) = (N_{d-1} - N_{d-1}/N_k)(i_k - j_k) \mod N_d.\]
If \(i_k > j_k\) then 
\[(N_{d-1} - N_{d-1}/N_k)(i_k - j_k) \mod N_d = N_d - N_{d-1}/N_k \geq N_{d-1} - N_{d-1}/N_q.\]
An important observation is that if \(q = d - 1\) we get an optimal lower bound of \(N_{d-1} - 1\).
If \(i_k < j_k\) then 
\[(N_{d-1} - N_{d-1}/N_k)(i_k - j_k) \mod N_d = N_d - (N_{d-1} - N_{d-1}/N_k)(j_k - i_k) \geq N_d - (N_{d-1} - N_{d-1}/N_k)(n_k - 1) = N_{d-1}(1 + 1/N_{k-1} - 1/N_k) > N_{d-1}.\]
c) Consider an edge of the \(k = d\)-th dimension. The analysis is the same as in Case I.b).
Case III. Let \(n_{d-1} = n_d\). We define a new labelling of the vertices of the Hamming graph which gives the alternative lower bound in Theorem 2.1.
\[f(i_1, i_2, \ldots, i_d) = ((N_{d-1} - N_0)i_1 + (N_{d-1} - N_1)i_2 + \ldots + (N_{d-1} - N_{d-2})i_{d-1} + N_{d-1}i_d) \mod N_d.\]
We show that \(f\) is a bijection. Assume that for two vertices \((i_1, i_2, \ldots, i_d) \neq (j_1, j_2, \ldots, j_d)\) 
\[f(i_1, i_2, \ldots, i_d) = f(j_1, j_2, \ldots, j_d).\]
It follows that 
\[(N_{d-1} - N_0)(i_1 - j_1) + \ldots + (N_{d-1} - N_{d-2})(i_{d-1} - j_{d-1}) + N_{d-1}(i_d - j_d) = 0 \mod N_d.\]
Observe that \(i_1 - j_1\) is divisible by \(n_1\). As \(|i_1 - j_1| < n_1\), we have \(i_1 = j_1\). Divide the congruence by \(n_1\). Then \(i_2 - j_2\) is divisible by \(n_2\). As \(|i_2 - j_2| < n_2\), we have \(i_2 = j_2\). Divide the congruence
by $n_2$ and continue in a similar way up to $i_d = j_d$, which is a contradiction. Distinguish 2 subcases.

a) Consider an edge of the $k$-th dimension, $k \leq d - 1$. Wlog assume that

\[ f(i_1, i_2, \ldots, i_k, \ldots, i_d) \geq f(i_1, i_2, \ldots, j_k, \ldots, i_d). \]

Then compute

\[ f(i_1, i_2, \ldots, i_k, \ldots, i_d) - f(i_1, i_2, \ldots, j_k, \ldots, i_d) = (N_{d-1} - N_{k-1})(i_k - j_k) \mod N_d. \]

If $i_k > j_k$ then

\[ (N_{d-1} - N_{k-1})(i_k - j_k) \mod N_d \geq N_{d-1} - N_{k-1} \geq N_{d-1} - N_{d-2}. \]

If $i_k < j_k$ then

\[ (N_{d-1} - N_{k-1})(i_k - j_k) \mod N_d \geq N_{d-1} - (N_{d-1} - N_{k-1})(j_k - i_k) \]

\[ \geq N_{d-1} - (N_{d-1} - N_{k-1})(n_k - 1) > N_{d-1}. \]

b) Consider an edge of the $d$-th dimension. The analysis is the same as in Case I.b). \hfill \square

For the cyclic antibandwidth of Hamming graphs we have (proof omitted):

**Theorem 2.2** For $d \geq 2$ and $2 \leq n_1 \leq n_2 \leq \ldots \leq n_d$,

\[ \text{cab}(\Pi_{k=1}^d K_{n_k}) = \text{ab}(\Pi_{k=1}^d K_{n_k}) \]

**Concluding Remarks.** To fill the gap in Theorem 2.1 remains a hard open problem. Note that the $d$-dimensional hypercube graph $Q_d$ is a special subcase. Recall that $\text{ab}(Q_d) = 2^{d-1}(1+o(1))$ was proved in [7]. Then $\text{ab}(Q_{d+1}) = 2^{d-1} - 2^d/\sqrt{2\pi d(1+o(1))}$ in [8] and finally $\text{ab}(Q_d) = 2^{d-1} - \sum_{m=0}^{d-2} \left( \frac{m}{\pi} \right)^{2m} \log(\frac{\pi}{4m})$ in [11]. An interesting open problem is the antibandwidth of $K_{n_1} \times K_{n_2} \times K_{n_3}$.

Theorem 2.1 says that the only remaining unclear case is $n_1 = n_2 = n_3 = n$. We conjecture that $\text{ab}(K_n \times K_n \times K_n) = n^2 - n$. For $d > 3$, we can prove that $\text{ab}(K_n^d) \leq n^{d-1} - \Theta(d^2)$ suggesting that the upper bound in Theorem 2.1 is strictly less than $n^{d-1} - 1$.

**References**


