Upward Geometric Graph Embeddings into Point Set

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T. Mchedlidze, A. Symvonis

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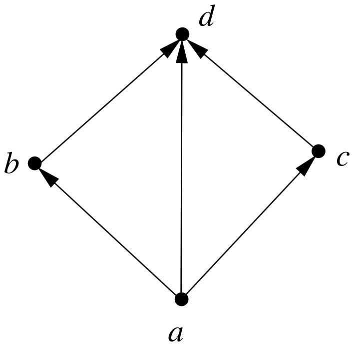
Problem Definition



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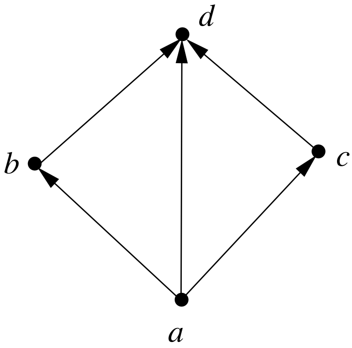

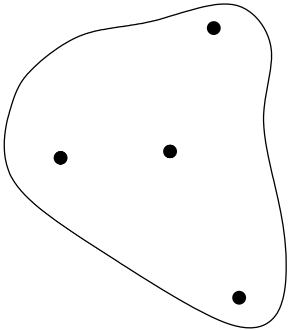

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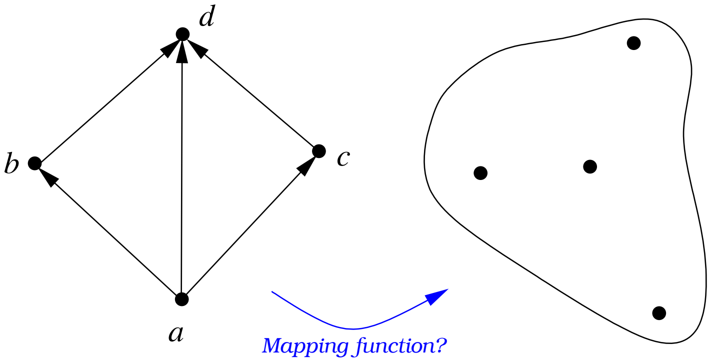

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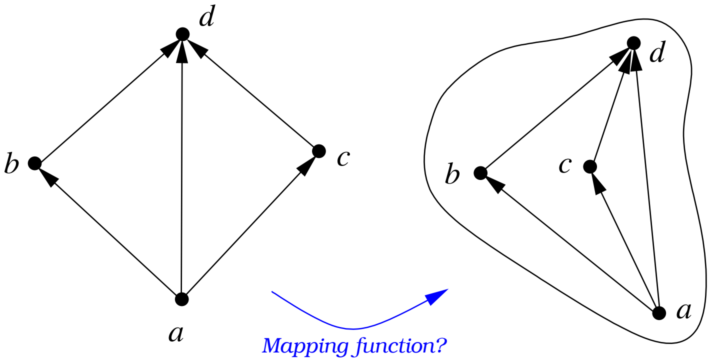

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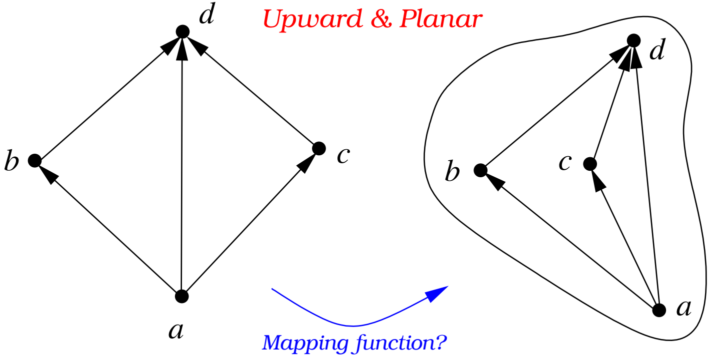


Previous Work

- Undirected graphs:
  - Studied from 1991
  - Outerplanar graphs, Gritzmann, Mohar, Pach, Pollack, AMM1991
  - Efficient algorithms to embed outerplanar graphs, trees Bose, GD97 and Bose, McAllister, Snoeyink, GD95
  - Decision problem is NP-Complete, Cabello, J. Graph Algorithms Appl,2006.

- Directed graphs:
  - Defined by Giacomo, Didimo, Liotta, Wismath, GD2002
  - Studied by:
    - Giordano, Liotta, Mchedlidze, Symvonis, ISAAC2007
    - Binucci, Giacomo, Didimo, Estrella-Balderama, Frati, Kobourov, Liotta, CompGeom:Theory&Appl. 2010
    - Angelini, Frati, Geyer, Kaufmann, Mchedlidze, Symvonis, GD2010

Summary of all known results on directed graphs

Graphs

	General	**One-Sided**	**Two-Sided**
Graphs	NP-Complete & Tightly Characterized	Characterized	**?**
Single source & cycle of length > 3	Not always & NP-Complete		
Single source & length of simple cycle 3	Always		
Two sources	**?**	Never	**?**

Summary of all known results on directed graphs

Trees

	General	**One-Sided**	**Two-Sided**
Trees	?	Always	Not Always, Polynomial (?)
Hourglass	Always		
Diameter 4	Always		
Caterpillars	?		Always
Switch	?		Always
k-Switch, $k \geq 2$			Not always

Summary of all known results on directed graphs

Paths

	General	**One-Sided**	**Two-Sided**
Path	?		Always
Every even monotone path has length one	Always		
One switch	Always		
Two switches & at least one monotone path has length one	Always		
Three switches & at least two monotone paths have length one	Always		

Outline

1. *The general problem is NP-complete*
2. *Switch Trees are embeddable to Convex Point Sets*
3. k*-Switch Trees are NOT embeddable to Convex Point Sets*

Upward embeddability is NP-hard

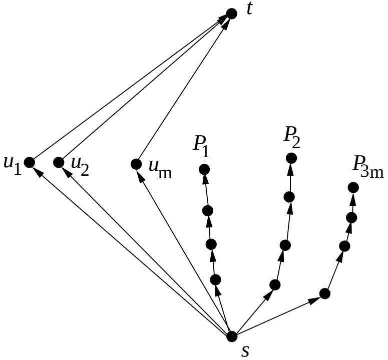
Problem: Given an upward planar digraph G, a general point set S, decide whether G is upward embeddable into S.

Theorem: The problem is *NP*-hard, even when

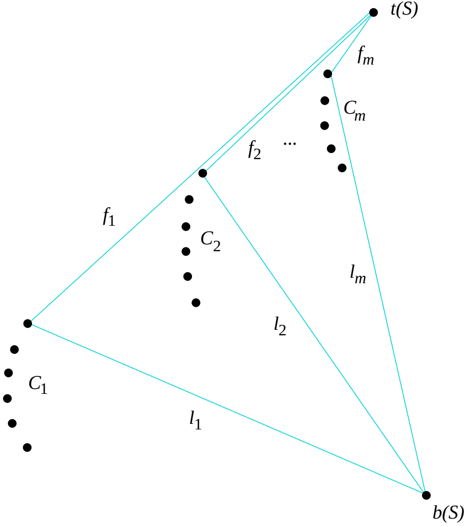
- G has a single source
- longest simple cycle of G has length at most 4
- S is an m-convex point set for some $m > 0$

Upward embeddability is NP-hard

Reduction from: 3-Partition.
Input: $B \in \mathbb{Z}^+$, and a set $A = \{a_1, \ldots, a_{3m}\}$
Output: $A_1, \ldots, A_m \subset A$ with $|A_i| = 3$ and $\sum_{a \in A_i} a = B$

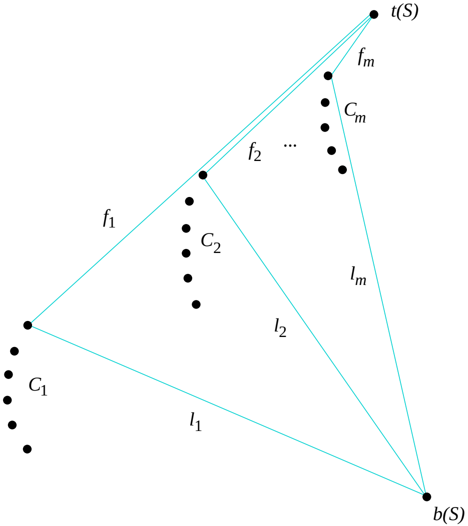


Upward embeddability is NP-hard



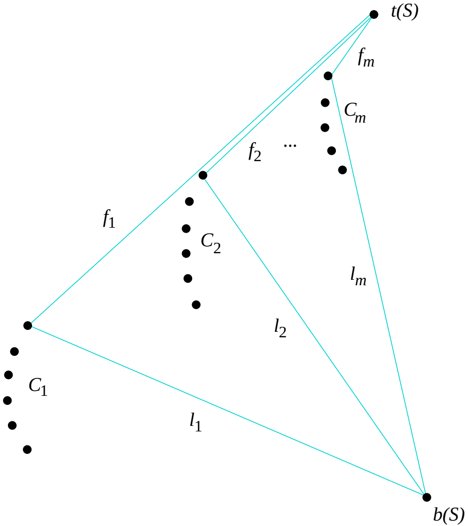
- $C_i \cup \{b(S), t(S)\}$ is a left-heavy convex point set

Upward embeddability is NP-hard



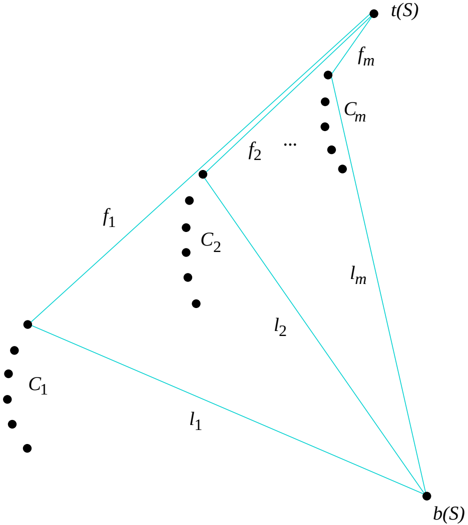
- C_{i+1} are higher than the points of C_i

Upward embeddability is NP-hard



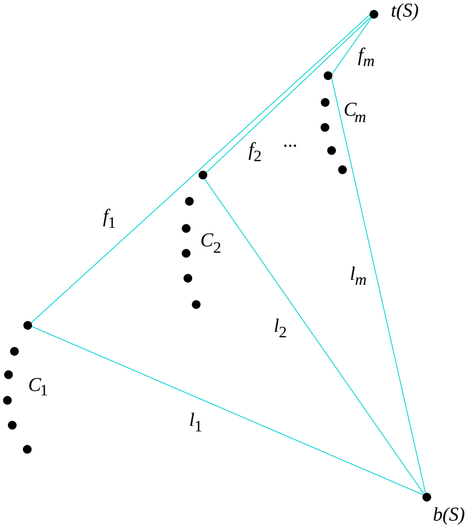
- $C_1, \ldots, C_i$ to the left of line l_i, $C_{i+1}, \ldots, C_m$ to the right of l_i

Upward embeddability is NP-hard



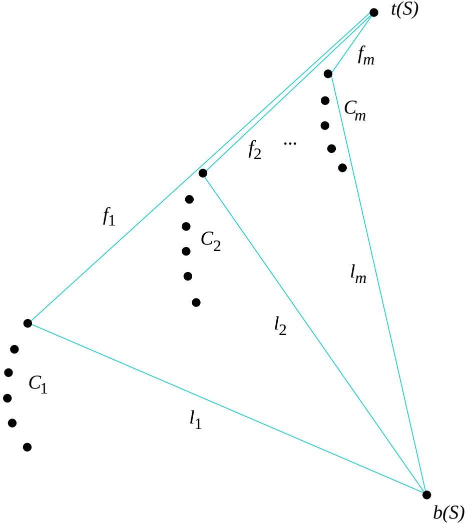
- C_j, $j \geq i$, to the right of f_i

Upward embeddability is NP-hard



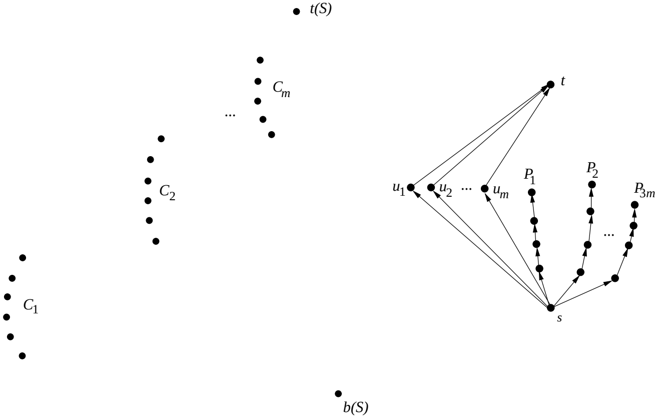
- $\{t(C_i) : i = 1, \ldots, m\}$ is a left-heavy convex point set

Upward embeddability is NP-hard



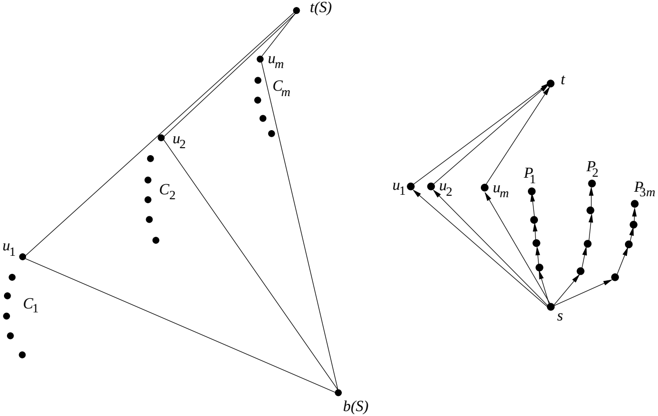
- Statement: S can be constructed polynomial on B and m.

Upward embeddability is NP-hard



[⇒] Assume that there is a 3-partition and construct a drawing.

Upward embeddability is NP-hard



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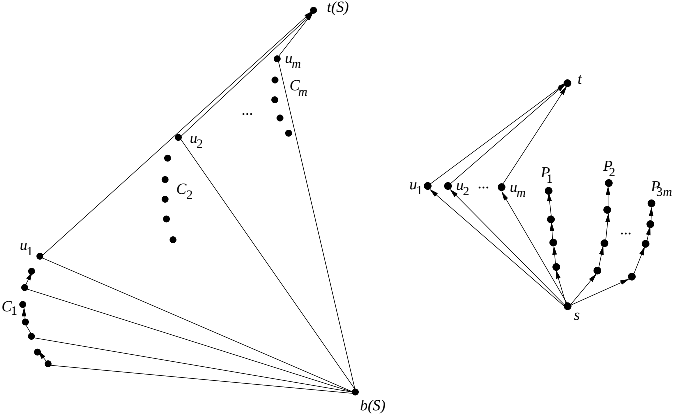

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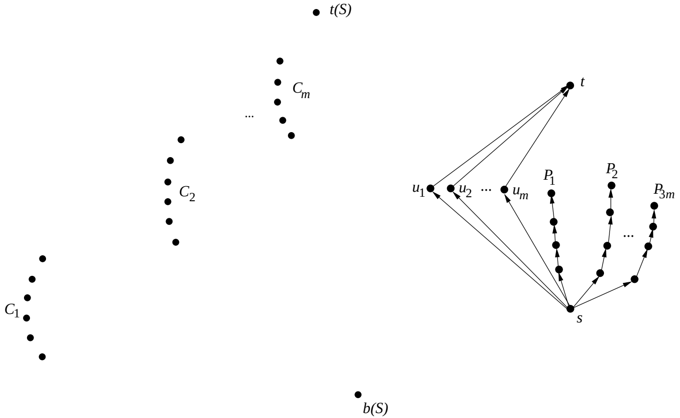
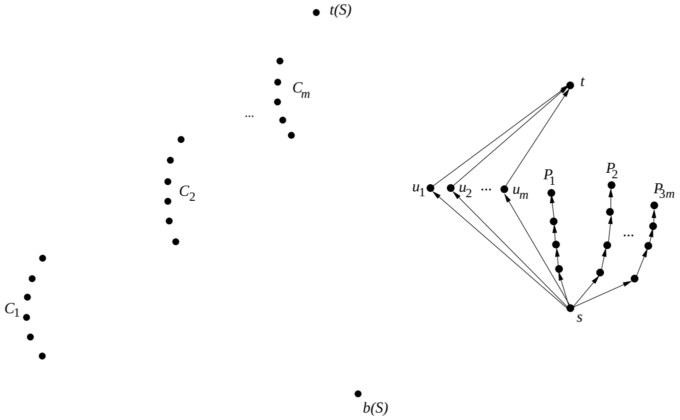


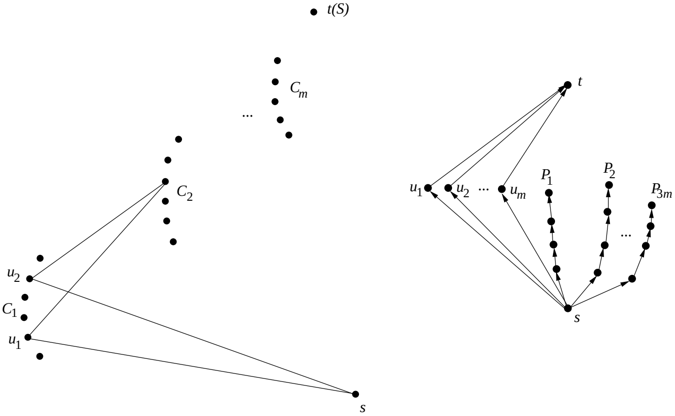
[⇐] Assume that there is a drawing and prove that there is a 3-Partition.

Upward embeddability is NP-hard



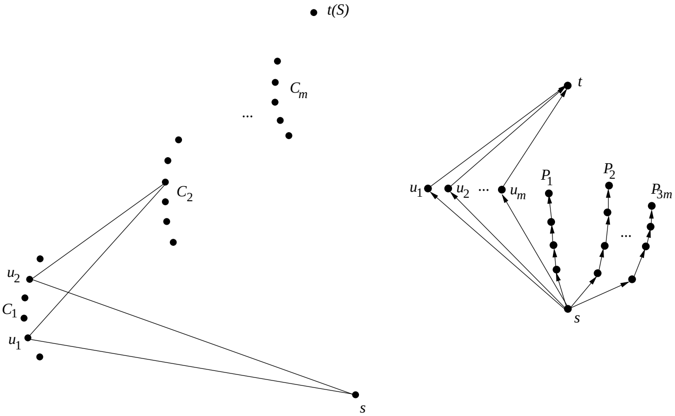
[⇐] Assume that there is a drawing then:
Fact 1: s is mapped to $b(S)$.

Upward embeddability is NP-hard



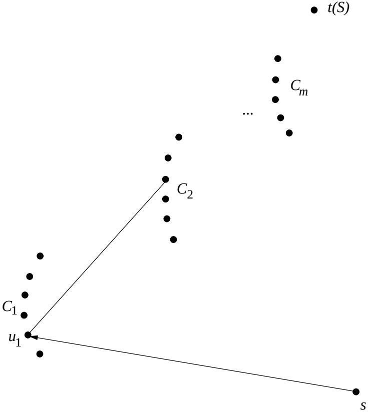
[⇐] Assume that there is a drawing then:
Fact 2: At most one u_i is mapped to C_j.

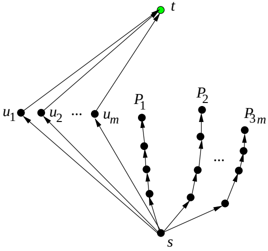
Upward embeddability is NP-hard



[⇐] Assume that there is a drawing then:
Fact 3: Vertex t is mapped either to $t(S)$ or to a point of C_m.

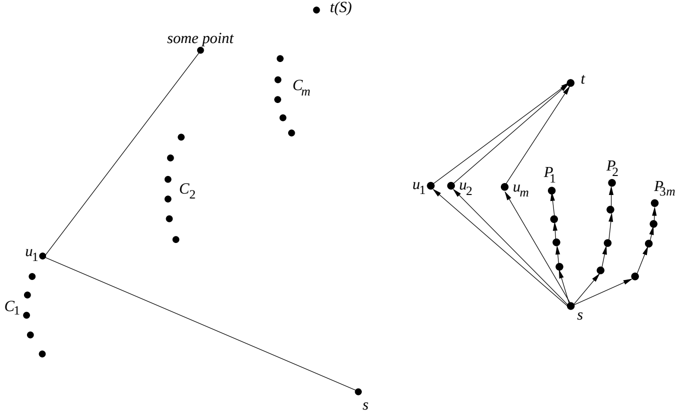
Upward embeddability is NP-hard





[⇐] Assume that there is a drawing then: Fact 4: Vertex u_i is mapped to $t(C_i)$ and there is no other arc outgoing from C_i.

Upward embeddability is NP-hard



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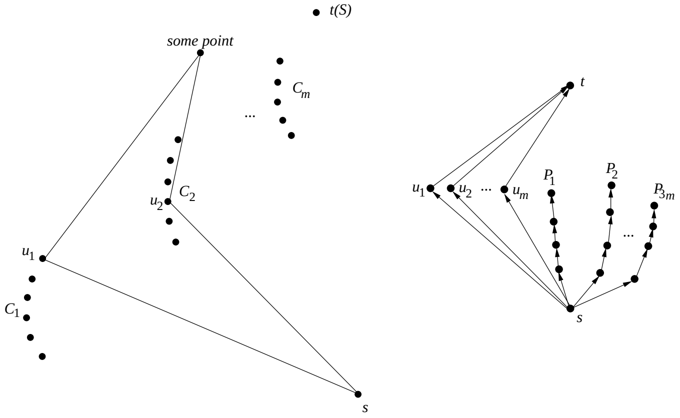

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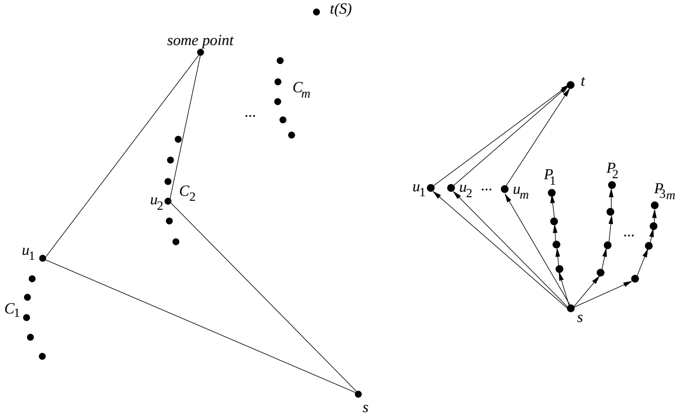
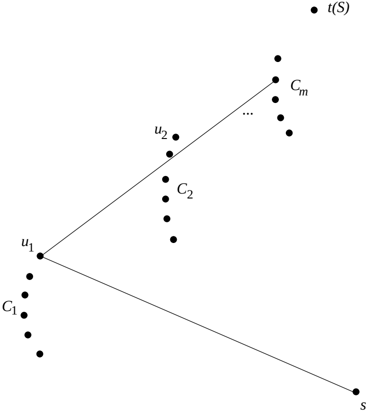
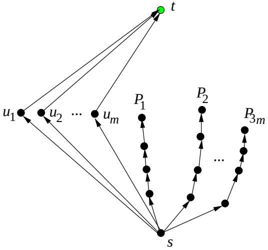


[⇐] Assume that there is a drawing then:
Fact 5: Each P_i is drawn entirely in some C_j.

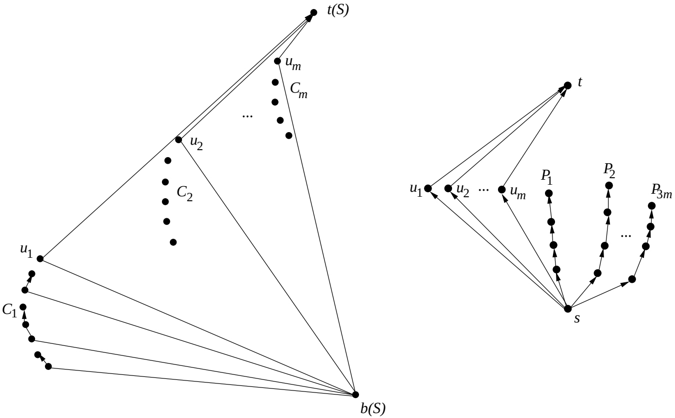
Upward embeddability is NP-hard





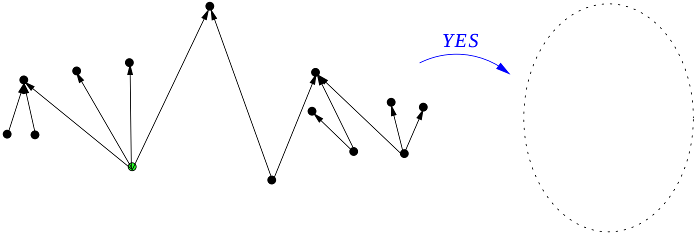
[⇐] Assume that there is a drawing then:
Fact 6: t is mapped to $t(S)$

Upward embeddability is NP-hard



[⇐] Assume that there is a drawing then:
We can construct a 3-Partition.

Switch trees



Theorem : Every switch tree is embeddable to any convex point set.

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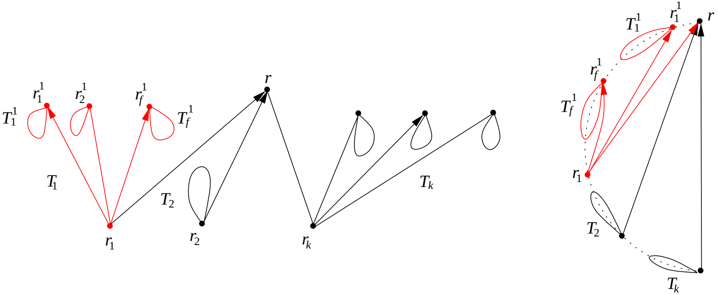
- One sided convex point set.



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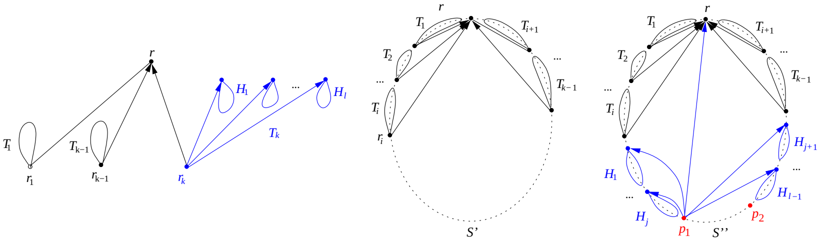
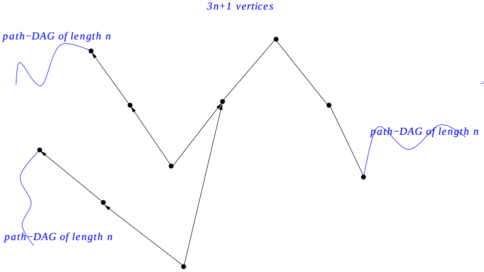
- Convex point set.

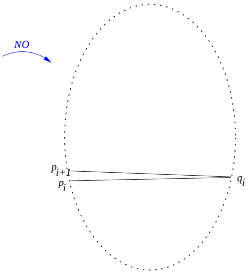


Generalize switch trees to k-switch trees.

Definition : Longest directed path has length at most k.

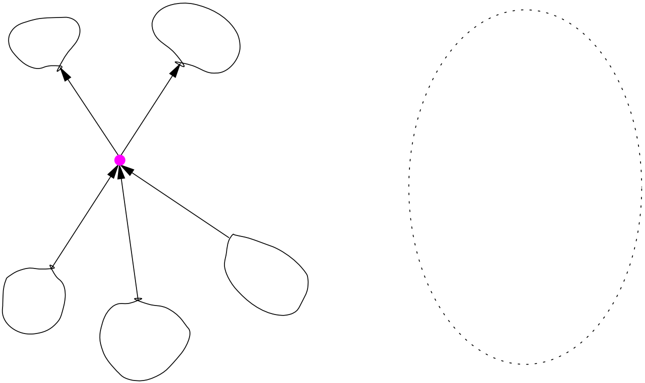
Theorem : For any $k \geq 2$ there are a lot k-switch trees that are not embeddable to some convex point set





k-switch trees are not embeddable to convex point sets

Lemma :[Binucci et al.]



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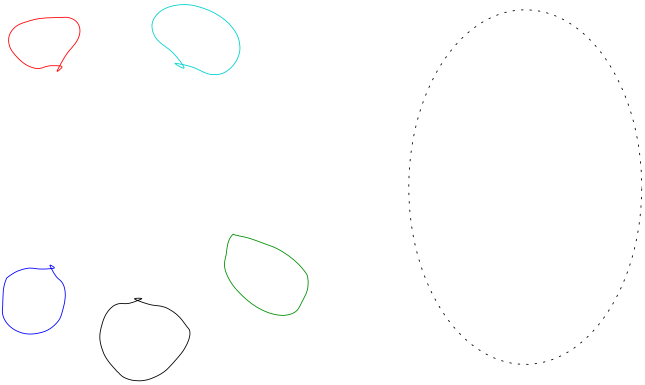

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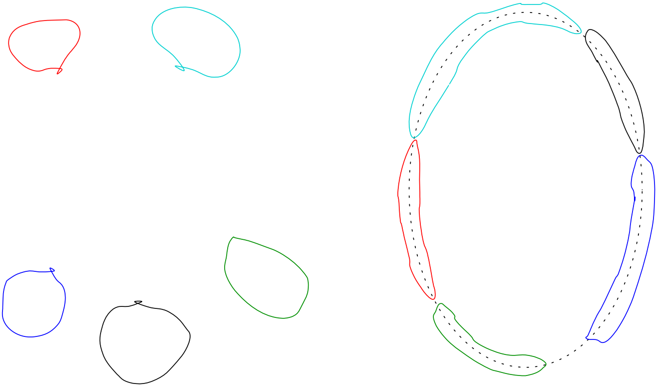

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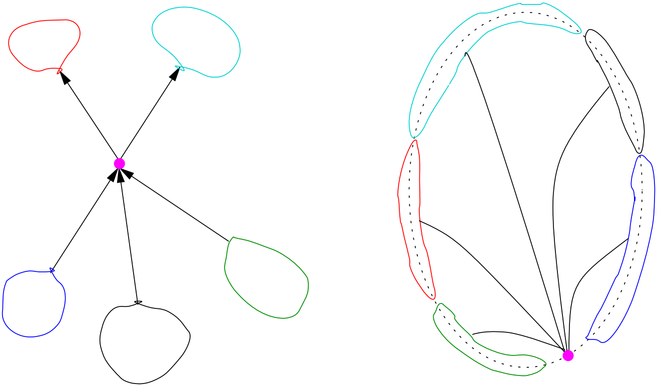
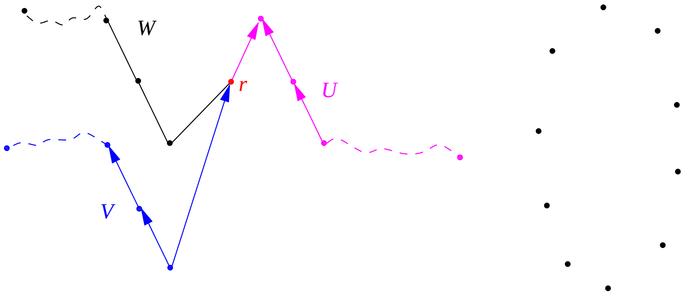


k-switch trees are not embeddable to convex point sets

- $|U| = |V| = |W| = n$
- Point set has size $3n + 1$
- Largest one-sided convex point subset $\lceil \frac{3n-1}{2} \rceil + 2 < 2n$
- At least one of U, V, W is mapped to a two-sided point set



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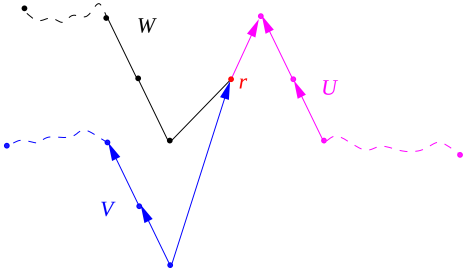

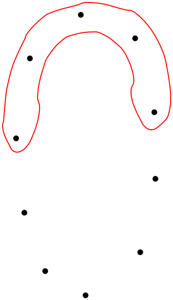

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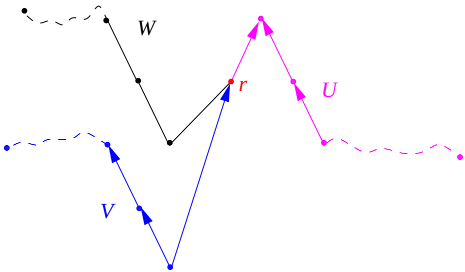

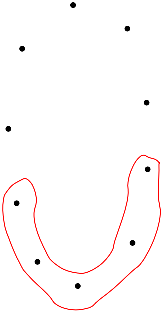

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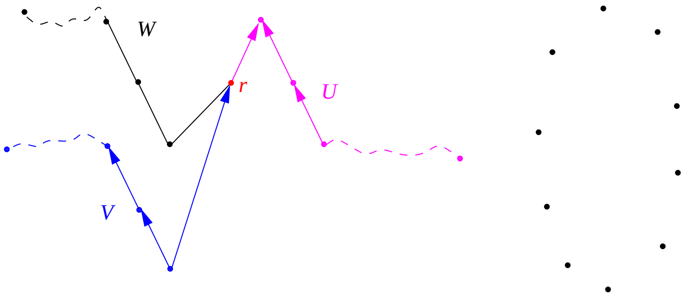

- Any upward drawing of U on a two-sided point set containing "top" create a crossing.
- Any upward drawing of V (W) on a two-sided point set containing "bottom" create a crossing. (symmetrically)
- There is no upward planar drawing of U on a two-sided point set containing "bottom".
- There is no upward planar drawing of V (W) on a two-sided point set containing "top". (symmetrically)

Open Problems

- Complexity of testing for:
  - Digraph/tree and convex point set?
  - Tree and general point set?
- Which other families of trees are embeddable to a convex point set?
- Is any path upward embeddable to general point set?
- Is there any real polynomial bound $f(n,k)$ on the number of points, so that any path with n vertices and k switched is upward embeddable to any point set with $f(n,k)$ points?

Thank You!