Finding the Description of a Structure by Counting Method: a Case Study

Ahti Peder and Mati Tombak

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A well known example

Fibonacci numbers are defined by a recurrent equation:

$$f_{n+1} = f_n + f_{n-1} \ (n \geq 1; f_0 = 0; f_1 = 1)$$

The beginning of the sequence is:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \ldots$$

Theorem

Closed formula for Fibonacci numbers is

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

The theorem can be proven by induction, but it is hard to imagine, how one can find the formula. Nevertheless, there is a well known method of generating functions which can be used for this kind of problems.

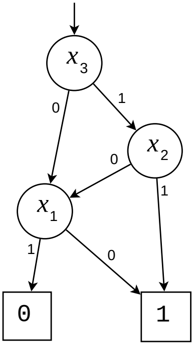
Binary graph

Definition

A *binary graph* is an oriented acyclic connected graph with root and two terminals (sinks), 0 and 1. Every internal node v has two sucessors: $high(v)$ and $low(v)$.

Therefore, an edge $a \to b$ is 0-*edge* (1-*edge*) if $low(a) = b$ ($high(a) = b$). Binary graphs are skeletons of binary decision diagrams (BDD): a BDD is a binary graph, in which internal nodes are labelled by propositional variables.

An example



BDD for a Boolean function $x_3 \& x_2 \vee \overline{x_1}$.

Superposition

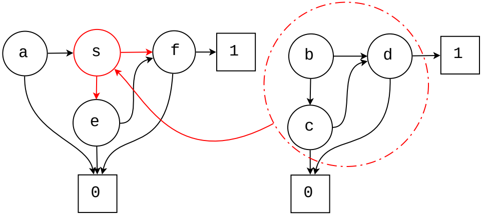
Definition

A *superposition* of a binary graph E into a binary graph G instead of an internal node v, denoted by $G_{v \leftarrow E}$, is a graph, which we get by

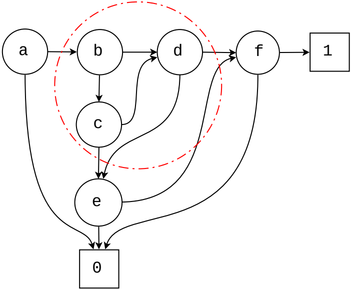
- deleting v from G,
- redirecting all edges, pointing to v, to the root of E,
- redirecting all edges of E pointing to terminal 1, to the node $high(v)$,
- redirecting all edges, pointing to the terminal 0, to the node $low(v)$.

An example

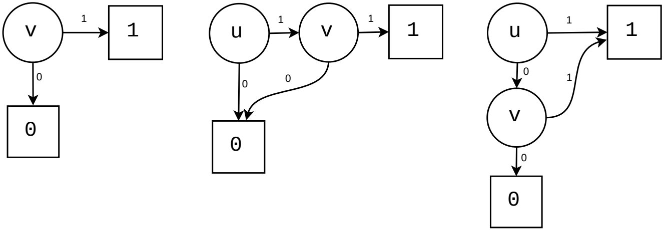
Binary graphs G ja E:



Binary graph $G_{s \leftarrow E}$:



Superpositional graph (SPG)



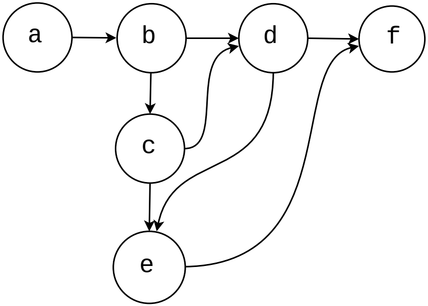
Elementary graphs A, C ja D.

Definition

1° $A \in SPG$;

2° if $G \in SPG$ and $v \in V(G)$, then $G_{v \leftarrow C} \in SPG$ and $G_{v \leftarrow D} \in SPG$.

An example



1-edges are directed from left to right and 0-edges from up to down.
Terminal nodes are omitted.

The problem

Problem

Find the description of superpositional graphs without using the notion of superposition.

Some properties

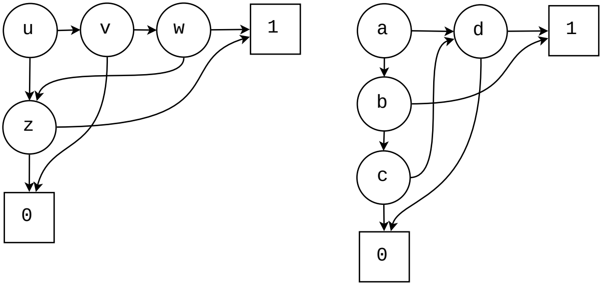
Theorem

Let $G \in SPG$. Then:

1. *G has a unique root;*
2. *G is planar;*
3. *G is acyclic;*
4. *there exists a directed path through all intermediate nodes and this Hamiltonian path is unique (we say that G is uniquely traceable);*
5. *G is homogenous (only one type of edges enters into every node $v \in V(G)$).*

Are these properties sufficient for a binary graph to be a superpositional?

Counterexamples



It is easy to see that all properties from previous theorem are fulfilled, but there does not exist a sequence of superpositions, generating these graphs.

Let us count!

- 1-node $\longrightarrow$ 1;
- 2-node $\longrightarrow$ 2;
- 3-node $\longrightarrow$ 6;
- 4-node $\longrightarrow$ 22;
- 5-node $\longrightarrow$ 90.

Are there mathematical structures described by sequence $1, 2, 6, 22, 90, \ldots$*?*
N. Sloane. On-Line Encyclopedia of Integer Sequences.
`http://www.research.att.com/~njas/sequences/` sequence A006318

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots$$

(Large Schroeder numbers)

The idea

Why not to try to express the properties of the binary graphs by propositional formulae?

It is straightforward to do, if we fix the number of internal nodes n. We are interested in properties for arbitrary binary graphs, so we should express the properties using family of formulae which depend on parameter n.

If some set of properties is formalized, then we can count the number of models for small values of n and compare with the sequence A006318

$$1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, \ldots$$

Propositional variables

Let the Hamiltonian path of an n-node SPG G consist of nodes $v_1, \ldots, v_n$; and let v_{n+1} represent both terminal nodes. Let

$$x_{i,j} = \begin{cases} 1, & \text{if there exists a 0-edge } v_i \rightarrow v_j; \\ 0, & \text{otherwise;} \end{cases}$$

$$y_{i,j} = \begin{cases} 1, & \text{if there exists a 1-edge } v_i \rightarrow v_j; \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$X = \{x_{i,j}, y_{i,j} : 1 \leq i < j \leq n+1\}$$

We are looking for a formula $\mathcal{F}_n^*(X)$, which should satisfy the condition

$$\mathcal{F}_n^*(\alpha) = \begin{cases} 1, & \text{if the graph represented by } \alpha \text{ is an SPG;} \\ 0, & \text{otherwise;} \end{cases}$$

for every assignment α to X.

Traceable binary graphs

- There is exactly one edge (0-edge or 1-edge) between the successive nodes of Hamiltonian path (except maybe between the last two nodes):

$$\mathcal{P}_1 = \bigwedge_{1 \le i \le (n-1)} xor(x_{i,i+1}, y_{i,i+1})$$

- Each node (except the last) has exactly one exiting 0-edge:

$$\mathcal{P}_2 = \bigwedge_{1 \le i \le n-1} exactlyone(x_{i,j} : i < j \le n+1)$$

- Each node (except the last) has exactly one exiting 1-edge:

$$\mathcal{P}_3 = \bigwedge_{1 \le i \le n-1} exactlyone(y_{i,j} : i < j \le n+1)$$

Numbers of traceable binary graphs: $1, 2, 8, 48, 384, 3840, \ldots$

The sequence should be: $1, 2, 6, 22, 90, 394, \ldots$

Tools

1. Translator, which translates a family of formulae into concrete propositional formula for given n (written by Ahti Peder in 2001).

2. Counter, which calculates the number of true assignments of the formula ($\sharp F$, written by Ain Isotamm in 1997).

Homogenous traceable binary graphs

- Every SPG is homogenous, i.e., 0-edges and 1-edges cannot enter the same internal node at the same time. As there exists also the Hamiltonian path, we can equally say that if a 0-edge $v_i \longrightarrow v_j$ exists, then the edge $v_{j-1} \longrightarrow v_j$ is a 0-edge:

$$\mathcal{P}_4 \equiv \bigwedge_{1 \leq i \leq (n-2)} \bigwedge_{(i+2) \leq j \leq n} (x_{i,j} \longrightarrow x_{j-1,j})$$

- Similarly, if there exists a 1-edge $v_i \longrightarrow v_j$, then the edge $v_{j-1} \longrightarrow v_j$ is a 1-edge:

$$\mathcal{P}_5 \equiv \bigwedge_{1 \leq i \leq (n-2)} \bigwedge_{(i+2) \leq j \leq n} (y_{i,j} \longrightarrow y_{j-1,j})$$

Numbers of homogenous binary traceable graphs:

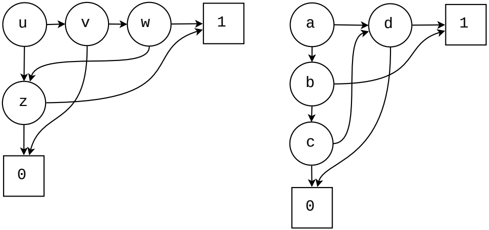
$$1, 2, 6, 24, 120, 720, \ldots.$$

The sequence should be:

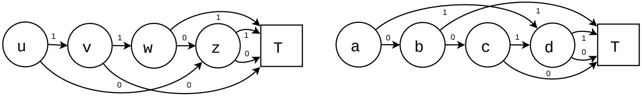
$$1, 2, 6, 22, 90, 394, \ldots.$$

The problematic assignments (graphs)

Analyzing the 24 assignments we received in case $n = 4$, we see that 2 of them do not represent any SPG:



We stretch the graphs so that all nodes are on straight line according to Hamiltonian path so that 1-edges are above and 0-edges below the line:



22 superpositional graphs did not have crossing edges in such drawing. We suppose that the edges of the same type should not cross.

Strong planarity

Let

$$1 \leq k < l < p < r \leq n+1$$

Definition

We say that edges $v_k \longrightarrow v_p$ and $v_l \longrightarrow v_r$ are *crossing* edges if $k < l < p < r$.

Definition

We say that a binary traceable graph is *strongly planar* if it has no crossing 0-edges and no crossing 1-edges.

$$\mathcal{P}_6 \equiv \bigwedge_{1 \leq k < l < p < r \leq (n+1)} (\overline{x_{k,p} \& x_{l,r}})$$

$$\mathcal{P}_7 \equiv \bigwedge_{1 \leq k < l < p < r \leq (n+1)} (\overline{y_{k,p} \& y_{l,r}})$$

The numbers of strongly planar graphs: $1, 2, 6, 22, 92, 422, \ldots$

The sequence should be: $1, 2, 6, 22, 90, 394, \ldots$

Intermediate hypotheses

Hypothesis

Every superpositional graph is strongly planar.

Additional experiments gave a surprising result: if we omit the conditions of homogeneity ($\mathcal{P}_4$ and $\mathcal{P}_5$), we receive the same sequence

$$1, 2, 6, 22, 92, 422, \ldots.$$

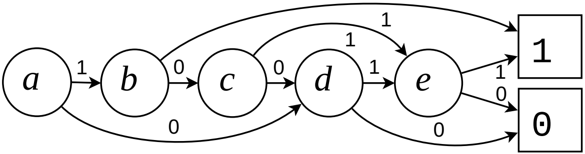
Hypothesis

Every strongly planar traceable binary graph is homogenous.

Analysis

Numbers of strongly planar graphs: $1, 2, 6, 22, 92, 422, \ldots$

The sequence should be: $1, 2, 6, 22, 90, 394, \ldots$ We analyzed the last approximation ($n = 5$) and found two problematic graphs:



The second graph is dual to the first.

We suppose that all 1-edges, starting within the endpoints of some 0-edge, must point to the same node.

The third hypothesis

Definition

We say that a binary traceable graph is 1-*cofinal (0-cofinal)* if all 1-edges (0-edges), starting between the endpoints of some 0-edge (1-edge) and crossing it, are entering into the same node.

Definition

We say that a binary traceable graph is *cofinal* if it is 1-cofinal and 0-cofinal.

Hypothesis

Every superpositional graph is cofinal.

Propositional formulae for cofinality

$$\mathcal{P}_8 \equiv \bigwedge_{1 \le k < s < l < p < r < t \le (n+1)} (\overline{x_{k,p} \& y_{l,r} \& y_{s,t}})$$

$$\mathcal{P}_9 \equiv \bigwedge_{1 \le k < s < l < p < r < t \le (n+1)} (\overline{y_{k,p} \& x_{l,r} \& x_{s,t}})$$

The propositional formula is

$$\mathcal{F} \equiv \mathcal{P}_1 \& \mathcal{P}_2 \& \mathcal{P}_3 \& \mathcal{P}_6 \& \mathcal{P}_7 \& \mathcal{P}_8 \& \mathcal{P}_9.$$

This formula gave us a sequence $1, 2, 6, 22, 90, 394, 1806, 8558, 41586, \ldots$ which is exactly the beginning of the sequence A006318 in N. Sloane. On-Line Encyclopedia of Integer Sequences.

The main hypothesis

Hypothesis

A binary graph is a superpositional graph if and only if it is a strongly planar cofinal traceable graph.

All four hypotheses have been proven in the paper:

Ahti Peder and Mati Tombak, Superpositional graphs. *Acta et Commentationes Universitatis Tartuensis de Mathematica*, 13, (2009), 51-64.

THANK YOU!

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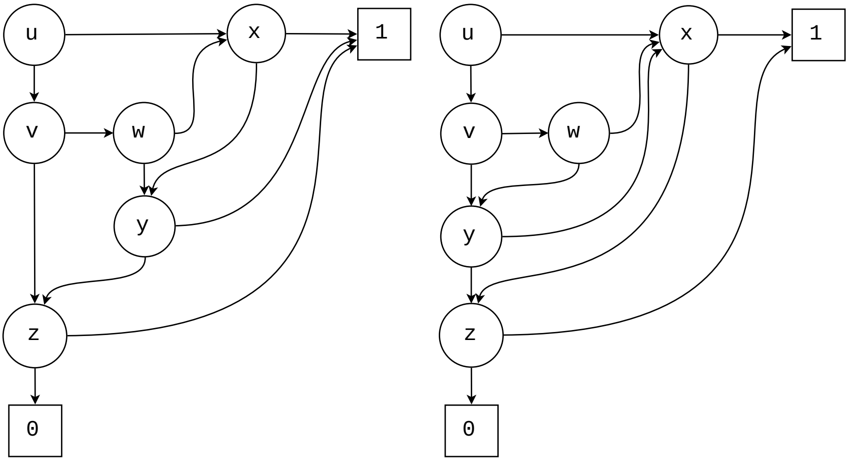
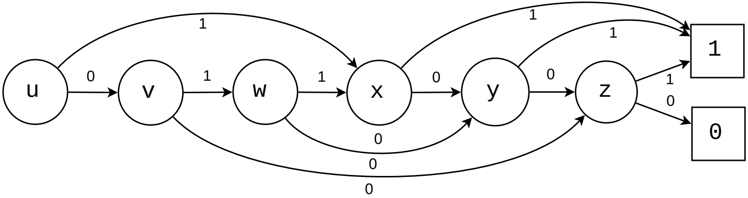
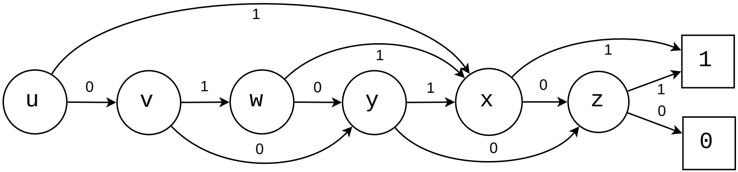
An application



Are these binary graphs superpositional?

An answer





THANK YOU!