On Semi-perfect 1-factorizations*

Rastislav Královič, Richard Královič

Department of Computer Science
Faculty of Mathematics, Physics and Informatics
Comenius University, Bratislava
Slovakia

Abstract. The *perfect 1-factorization conjecture* by A. Kotzig [7] asserts the existence of a 1-factorization of a complete graph K_{2n} in which any two 1-factors induce a Hamiltonian cycle. This conjecture is one of the prominent open problems in graph theory. Apart from its theoretical significance it has a number of applications, particularly in designing topologies for wireless communication. Recently, a weaker version of this conjecture has been proposed in [1] for the case of *semi-perfect 1-factorizations*. A semi-perfect 1-factorization is a decomposition of a graph G into distinct 1-factors $F_1, \ldots, F_k$ such that $F_1 \cup F_i$ forms a Hamiltonian cycle for any $1 < i \leq k$. We show that complete graphs K_{2n}, hypercubes Q_{2n+1} and tori $T_{2n\times 2n}$ admit a semi-perfect 1-factorization.

1 Introduction

In this paper we deal with 1-factorizable graphs, i.e. graphs whose edges can be decomposed into 1-factors (perfect matchings). Clearly, taking the union of any two 1-factors F_i and F_j gives a 2-factor: a spanning subgraph consisting of a set of vertex-disjoint cycles. Additionally, if $F_i \cup F_j$ is connected it forms a Hamiltonian cycle and the corresponding 1-factors F_i, F_j are said to form a *perfect pair*.

It is widely known that a complete graph K_{2n} is 1-factorizable, see e.g. [8] for a survey. In his 1963 paper [7], A. Kotzig conjectured that for every $n \geq 2$ the complete graph K_{2n} can be decomposed into $n-1$ one-factors in such a way that any two of them form a perfect pair. Despite an extensive effort, this conjecture is still open. Currently it is known that such *perfect factorization* exists if either n is prime, $2n-1$ is prime or $2n \in \{16, 28, 36, 40, 50, 126, 170, 244, 344, 730, 1332, 1370, 1850, 2198, 3126, 6860\}$ (the references can be found in [10]).

One possible application of the perfect 1-factorization comes from the area of wireless communication. In [3] the problem of building a topology for an ad-hoc network of Bluetooth devices is addressed. As each device can communicate with exactly one other device at a time, the communication pattern at a given time forms a matching. In a bandwidth-efficient topology, a number k of 1-factors is used for communication in a time-multiplexed fashion, where k is

* Supported in part by grant APVT-20-018902

the parameter of the network: larger values of k increase the robustness and decrease the diameter of the network while introducing more communication overhead due to interference. To achieve fairness, the chosen 1-factors should cover all edges. Moreover, any two of them should form a connected graph. In [3], a 1-factorization of K_{2n} into $F_1, \ldots, F_{n-1}$ is presented, such that any two 1-factors from $\{F_1, \ldots, F_p\}$ form a perfect pair, where p is the smallest prime factor of $2n-1$. Hence, the possible choices of the parameter k depend on the number of vertices which is not very convenient.

As a step towards the general solution, [1, 6] propose to solve a weaker version of the conjecture: find a 1-factorization of a graph G into 1-factors $F_1, \ldots, F_k$ such that $F_1 \cup F_i$ forms a Hamiltonian cycle for any $1 < i \leq k$. Such a 1-factorization will be called *semi-perfect*.

The semi-perfect 1-factorization conjecture has application also to the topological graph theory. In [5], the genus of joins and compositions of graphs is studied. According to [1]:

> *This kind of edge coloring* (i.e. semi-perfect 1-factorization) *of the cubes (and graphs in general) would lead to an improvement of existing genus results for joins and compositions of these graphs...*

In this paper we show that complete graphs K_{2n}, hypercubes Q_{2n+1} and tori $T_{2n\times 2n}$ admit a semi-perfect 1-factorization.

The potential of semi-perfect 1-factorizations in the topology design is yet to be investigated.

2 Preliminaries

Unless stated otherwise, we consider simple undirected graphs. A 1-factor (i.e. perfect matching) of a graph G is a spanning subgraph in which all vertices have degree 1. A 1-factorization is a decomposition of the set of edges into distinct 1-factors.

A 1-factorization of a graph G into 1-factors $F_1, \ldots, F_k$ is called *semi-perfect* if $F_1 \cup F_i$ forms a Hamiltonian cycle (i.e. connected 2-factor) for any $1 < i \leq k$.

Instead of constructing the particular 1-factors F_i, we shall construct the respective Hamiltonian cycles $F_1 \cup F_i$ according to the following definition:

Definition 1. *Let G be a graph, P be a 1-factor of G and $H_1, \ldots, H_k$ be Hamiltonian cycles of G such that each H_i contains P as its subset. Furthermore, let each edge of $G \setminus P$ be covered by exactly one cycle H_i. We call the set $\{H_1, \ldots, H_k\}$ a P-*cover *of G.*

It is easy to see that from given G, P and $H_1, \ldots, H_k$ that form a P-cover of G we can construct a semi-perfect 1-factorization of G. Indeed, let $P_1 := P$ and $P_{i+1} := H_i \setminus P$ for each $1 \leq i \leq k$. The 1-factors $P_1, \ldots, P_{k+1}$ form a semi-perfect 1-factorization of G.

In order to construct semi-perfect 1-factorizations of complete graphs and hypercubes we shall investigate the following two graph operations:

Definition 2. *Let $G = (V, E)$ be a graph. We define a graph $O_1(G)$ as follows: Each vertex v is replaced by vertices v_0 and v_1 connected by an edge. Each edge (u, v) is replaced by edges (u_0, v_0), (u_1, v_1), (u_0, v_1), (u_1, v_0). Formally,*

$$O_1(G) = (V', E'), \quad V' = \{v_0, v_1 \mid v \in G\}$$

$$E' = \{(v_0, v_1) \mid v \in G\} \cup \{(u_0, v_0), (u_1, v_1), (u_0, v_1), (u_1, v_0) \mid (u, v) \in E\}$$

Similarly, we define $O_2(G)$ as

$$O_2(G) = (V', E'), \quad V' = \{v_0, v_1 \mid v \in G\}$$

$$E' = \{(v_0, v_1) \mid v \in G\} \cup \{(u_0, v_0), (u_1, v_1) \mid (u, v) \in E\}$$

The graph $O_2(G)$ is the usual Cartesian product $G \times K_2$.

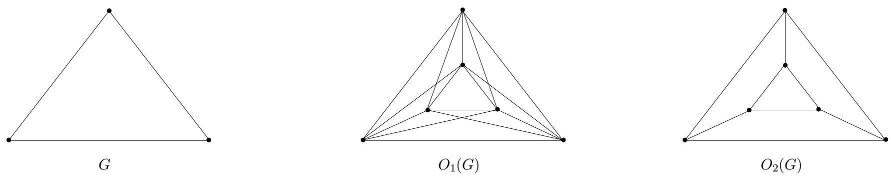

Fig. 1. The operations $O_1(G)$ and $O_2(G)$

3 Complete Graphs

In this section we show that complete graphs K_{2n} admit a semi-perfect 1-factorization. To do so we use the facts that $K_{2n} \cong O_1(K_n)$ and the directed complete graph K_n^* has a Hamiltonian decomposition, i.e. the arcs of K_n^* can be decomposed into disjoint Hamiltonian cycles.

Lemma 1. *Let G be an undirected graph without loops and multiple edges and G^* be a directed graph obtained from G by replacing each edge by two opposite arcs. Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a set of Hamiltonian cycles on G^*, such that each arc of G^* is covered exactly once by $\mathcal{H}$. Let $G' = O_1(G)$ and $P = \{(v_0, v_1) \mid v \in V(G)\}$ be a 1-factor of G'. Then there exists a P-cover of G', i.e. a semi-perfect 1-factorization of G'.*

Proof. We shall create the P-cover $\mathcal{H}' = \{H'_1, \ldots, H'_{2k}\}$ of the graph G' by constructing two Hamiltonian cycles H'_{2i-1}, H'_{2i} of G from each Hamiltonian cycle H_i.

Without loss of generality let $V(G) = \{1, \ldots, n\}$. Let $H_i = \{u_1, \ldots, u_n\}$, where $u_1 = 1$. Every second edge of the cycles H'_{2i-1} and H'_{2i} will be from P, i.e. of the form $\{x_0, x_1\}$ for some x in $V(G)$. We construct a Hamiltonian cycle

H'_{2i-1} as a sequence of vertices $(v_1, \ldots v_{2n})$, such that for the $2j-1$-st edge of the sequence, denoted e_{2j-1}, it holds $e_{2j-1} = \{v_{2j-1}, v_{2j}\} = \{x_0, x_1\}$ where $x = u_j$. What remains to be set is the orientation of the edge e_{2j-1} in the sequence, i.e. whether $v_{2j-1} = x_0$ and $v_{2j} = x_1$ or vice versa.

We say that the edge e_{2j-1} is oriented positively if $v_{2j-1} = x_0 \wedge v_{2j} = x_1$ and negatively if $v_{2j-1} = x_1 \wedge v_{2j} = x_0$. Define the orientation of e_{2j-1} inductively in the following way: Edge e_1 is oriented positively, i.e. $v_1 = u_{10} = 1_0$ and $v_2 = u_{11} = 1_1$. If $u_{j-1} < u_j$ then edge e_{2j-1} has the same orientation as the edge e_{2j-3}, otherwise it has opposite orientation.

The Hamiltonian cycle H'_{2i} is constructed in the same way as H'_{2i-1}, except that orientation of edges e_{2j-1} for all $j > 1$ is flipped.

From the construction of $\mathcal{H}'$ it follows that each H'_i contains the matching P as its subset. It remains to show that each edge of $E(G') \setminus P$ is covered by exactly one H'_i.

Let $\{u, v\}$ be any edge of G such that $u < v$. Let H_x be the only member of $\mathcal{H}$ that covers the edge (u, v) and H_y be the only member of $\mathcal{H}$ that covers the edge (v, u). This situation is illustrated on Figure 2.

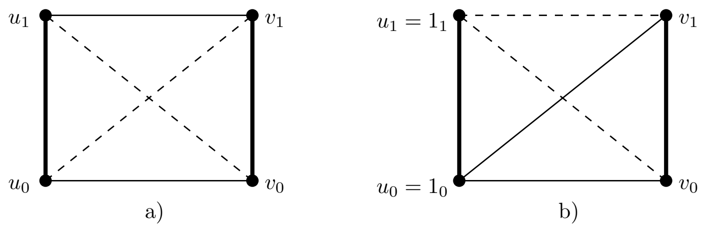

Fig. 2. Construction used in Lemma 1 where $u < v$, edge (u, v) is covered by H_x and edge (v, u) is covered by H_y. Dashed lines belong to cycles H'_{2x-1} and H'_{2x}. Thin solid lines belong to cycles H'_{2y-1} and H'_{2y}. The case $u > 1$ is presented on the left-hand side. The case $u = 1$ is presented on the right-hand side.

Assume that $u > 1$. Obviously $x \neq y$, hence H_x and H_y contribute to four different Hamiltonian cycles on G': H'_{2x-1}, H'_{2x}, H'_{2y-1}, and H'_{2y}. The construction of $\mathcal{H}'$ ensures that edges $\{u_0, u_1\}$ and $\{v_0, v_1\}$ have the same orientation in H'_{2x-1} and H'_{2x} and opposite orientation in H'_{2y-1} and H'_{2y}. Thus it holds that edges $\{u_1, v_0\}$ and $\{u_0, v_1\}$ are covered only by H'_{2x-1} and H'_{2x} and edges $\{u_0, v_0\}$ and $\{u_1, v_1\}$ are covered only by H'_{2y-1} and H'_{2y}.

Now consider the case $u = 1$. Clearly the edges $\{1_1, v_0\}$, $\{1_1, v_1\}$ are covered only by the cycles H'_{2x-1} and H'_{2x}. Similarly, the edges $\{1_0, v_0\}$, $\{1_0, v_1\}$ are covered only by the cycles H'_{2y-1} and H'_{2y}. □

Now we plug the facts about complete graphs into the preceding lemma:

Theorem 1. *Let $n > 1$ and K_{2n} be a complete graph with $2n$ vertices. Then there exists a semi-perfect 1-factorization of K_{2n}.*

Proof. Obviously, the graph $K_{2n} \cong O_1(K_n)$. Assume that $n \notin \{4,6\}$. By the result of Tillson [11] it is known that an oriented complete graph K_n^* is decomposable into $n-1$ Hamiltonian cycles for all $n \notin \{4,6\}$. Hence according to Lemma 1 there exists a semi-perfect 1-factorization of K_{2n}.

For cases $n=4$ and $n=6$ we use similar approach as in Lemma 1: take the appropriate $n-1$ Hamiltonian cycles $\mathcal{H}$ on K_n and construct $2n-2$ Hamiltonian cycles on $O_1(K_n)$ that form a P-cover. Since $\mathcal{H}$ is not a decomposition of K_n^*, the technique for assigning orientations to e_j from Lemma 1 does not work. However, it is possible to find a suitable assignment of orientations.

For $n=4$ we take Hamiltonian cycles $H_1=(1,2,3,4)$, $H_2=(1,3,4,2)$ and $H_3=(1,4,2,3)$. The resulting semi-perfect 1-factorization of K_8 is shown on Figure 3.

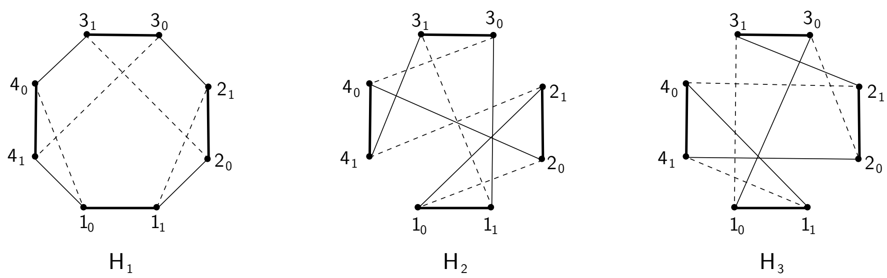

Fig. 3. A semi-perfect 1-factorization of $O_1(K_4) \cong K_8$.

For $n=6$ we take Hamiltonian cycles $H_1=(1,2,6,3,4,5)$, $H_2=(1,6,4,2,5,3)$, $H_3=(1,5,4,3,2,6)$, $H_4=(1,3,6,5,2,4)$ and $H_5=(1,4,6,5,3,2)$. The resulting semi-perfect 1-factorization of K_{12} is shown on Figure 4. □

4 Hypercubes

To show that hypercubes Q_{2n+1} admit a semi-perfect 1-factorization we use the same technique as for complete graphs. We use the fact that $Q_n \cong O_2(Q_{n-1})$ and that an even hypercube is Hamiltonian decomposable.

Lemma 2. *Let G be an undirected graph without loops and multiple edges with an even number of vertices. Let $\mathcal{H}=\{H_1,\dots,H_k\}$ be a set of Hamiltonian cycles on G, such that each edge of G is covered exactly once by $\mathcal{H}$. Let $G'=O_2(G)$ and $P=\{(v_0,v_1) \mid v \in V(G)\}$ be a matching of G'. Then there exists a P-cover of G', hence there exists a semi-perfect 1-factorization of G'.*

Proof. We shall create the P-cover $\mathcal{H}'=\{H'_1,\dots,H'_{2k}\}$ of the graph G' by constructing two Hamiltonian cycles H'_{2i-1}, H'_{2i} on G from a Hamiltonian cycle H_i in a similar fashion as in Lemma 1. However, since edges $\{u_0,v_1\}$ and $\{u_1,v_0\}$

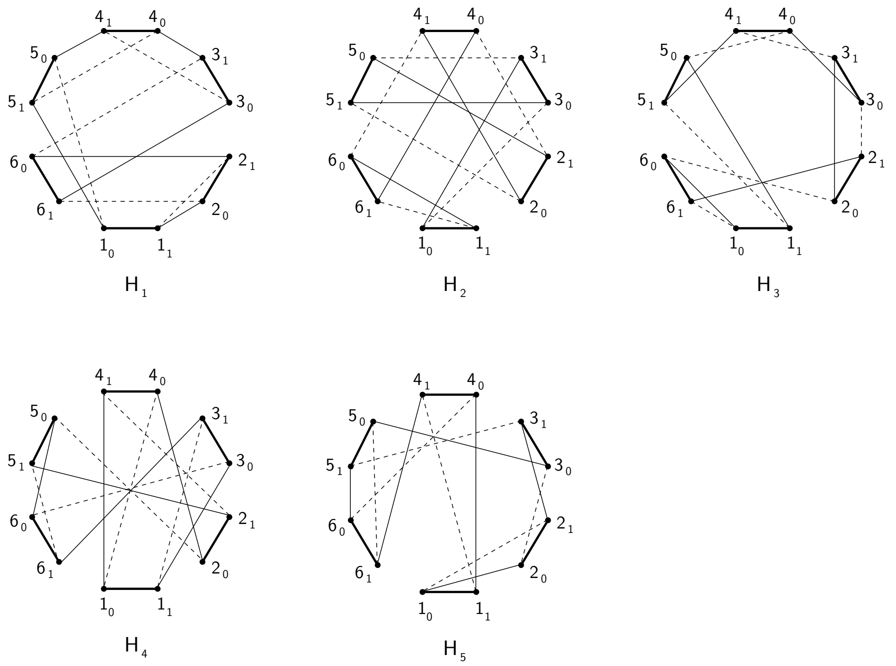

Fig. 4. A semi-perfect 1-factorization of $O_1(K_6) \cong K_{12}$.

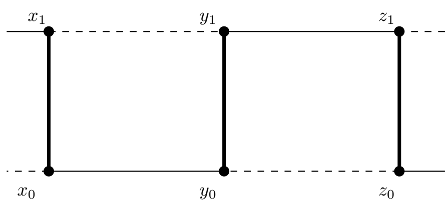

Fig. 5. Construction used in Lemma 2 where $x = u_{2j-1}$, $y = u_{2j}$ and $z = u_{2j+1}$ for some $H_i = (u_1, \ldots, u_{2m})$. Dashed lines belong to the cycle H'_{2i-1}. Thin solid lines belong to the cycle H'_{2i}.

do not exist in G', the orientation of edges in the constructed cycles must be alternating. To ensure the correctness of the construction, the number of vertices of G must be even.

Let $n = 2m$, $V(G) = \{1, \ldots, 2m\}$ and $H_i = (u_1, \ldots, u_{2m})$, where $u_1 = 1$. We construct a Hamiltonian cycle H'_{2i-1} as a sequence of vertices $(v_1, \ldots, v_{4m})$, such that $v_{4j-3} = {u_{2j-1}}_0$, $v_{4j-2} = {u_{2j-1}}_1$, $v_{4j-1} = {u_{2j}}_1$ and $v_{4j} = {u_{2j}}_0$.

The Hamiltonian cycle H'_{2i} is constructed in the same way as H'_{2i-1}, except that the orientation of all edges is flipped: $v_{4j-3} = {u_{2j-1}}_1$, $v_{4j-2} = {u_{2j-1}}_0$, $v_{4j-1} = {u_{2j}}_0$ and $v_{4j} = {u_{2j}}_1$. This construction is illustrated by Figure 5.

Obviously each H'_i contains the matching P as its subset. Any edge of $E(G') \setminus P$ can be expressed as $\{u_q, v_q\}$ for some $q \in \{0, 1\}$. Let H_i be the only member of $\mathcal{H}$ that covers the edge $\{u, v\}$. The construction of $\mathcal{H}'$ ensures that the edge $\{u_q, v_q\}$ is covered by either H'_{2i-1} or H'_{2i}. Hence $\mathcal{H}'$ is a P-cover of the graph G'. □

Theorem 2. *Let $n \geq 1$ and Q_{2n+1} be a hypercube with 2^{2n+1} vertices. Then there exists a semi-perfect 1-factorization of Q_{2n+1}.*

Proof. It is easy to see that the hypercube Q_{2n+1} is isomorphic to $O_2(Q_{2n})$. By the result of Aubert and B. Schneider [2] (see also [9, 4]) it is known that a hypercube Q_{2n} is decomposable into n Hamiltonian cycles for all n. Hence according to Lemma 2 there exists a semi-perfect 1-factorization of Q_{2n+1}. □

5 Tori

In this section we show that a torus T_{2n} of size $2n \times 2n$ admits a semi-perfect factorization. Obviously, torus T_n where n is odd contains an odd number of vertices and thus does not admit a semi-perfect 1-factorization.

We shall denote the vertices of a torus T_n by pairs $[i, j]$, $i, j \in \{1, \ldots, n\}$, such that two vertices $[i, j]$ and $[k, l]$ are connected by an edge if and only if exactly one of the two conditions holds: $j = l \wedge |k-i| \in \{1, n-1\}$ or $k = i \wedge |j-l| \in \{1, n-1\}$. Edges with $|k - i| = n - 1$ or $|j - l| = n - 1$ are called wrap-around edges.

Since tori are 4-regular, a 1-factorization can be viewed as an edge coloring using 4 colors. This 1-factorization is semi-perfect if and only if edges of color 0 together with edges of any other color form a Hamiltonian cycle.

It is possible to embed a torus T_n in an infinite grid by cutting the wrap-around edges. Edges $\{[i, 1], [i, n]\}$ are replaced by edges $\{[i, 0], [i, 1]\}$ and $\{[i, n], [i, n+1]\}$; edges $\{[1, i], [n, n]\}$ are replaced by edges $\{[0, i], [1, i]\}$ and $\{[n, i], [n+1, i]\}$. By this operation we obtain an *embedded torus ET_n*.

Definition 3. *An* embedded torus ET_n *is a subgraph of infinite two-dimensional grid induced by vertices $\{1, \ldots, n\} \times \{1, \ldots, n\}$ united with the set of* external edges E_{ext} *defined as follows:*

$$E_{ext} = \big\{\{[0, i], [1, i]\},\ \{[n, i], [n+1, i]\},\ \{[i, 0], [i, 1]\},\ \{[i, n], [i, n+1]\} \mid 1 \leq i \leq n\big\}$$

Edges of ET_n that are not external are called internal edges.

A Hamiltonian cycle in a torus may form several paths in the corresponding embedded torus. All these paths are terminated by an external edge. We call these paths *partial paths*.

Definition 4. *Let ET be an embedded torus with a given edge 4-coloring. A path containing only edges of color 0 and color i for some fixed i is called a* partial path *if and only if it starts and ends with an external edge.*

We show that it is possible to color the edges of an embedded torus by 4 colors such that certain invariant about the partial paths is preserved.

Lemma 3. *Let ET_n be an embedded torus of size $n \times n$, such that $n \geq 4$ is even. Then it is possible to color its edges in such a way that the external edges are colored as in Figure 6. Moreover, each edge is covered by some partial path and the external edges are connected by partial paths according to Figure 7. Exact formulation of these invariants is presented in the Appendix.*

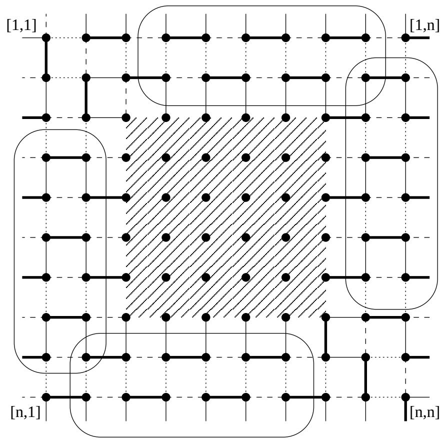

Fig. 6. Edge 4-coloring of an embedded torus used in Lemma 3. Color 0 is printed as bold lines, color 1 as thin lines, color 2 as dotted lines, and color 3 as dashed lines. Regularly repeating patterns are marked by ellipses. Formal description of this coloring is presented in the Appendix.

Proof. We shall construct the described coloring of ET_n inductively. The appropriate coloring for $n = 4$ and $n = 6$ is presented on Figure 8.

Now assume that $n \geq 8$ and we already have an appropriate coloring for an embedded torus ET' of size $n - 4 \times n - 4$. We insert the torus ET' into the embedded torus ET of size $n \times n$ such that a vertex $[i, j]$ is mapped onto the vertex $[i + 2, j + 2]$, hence vertices of the torus ET' are mapped onto vertices $\{3, \ldots, n-2\} \times \{3, \ldots, n-2\}$. Edges not covered by ET' (i.e. edges incident to $V(ET) \setminus V(ET'))$ are colored according to Figure 6. It is straightforward to check

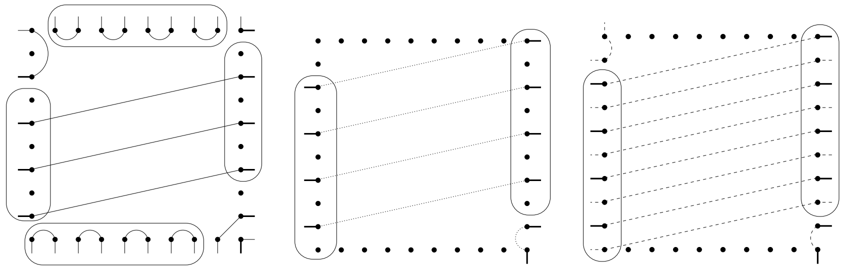

Fig. 7. Connectivity of partial paths used in Lemma 3. Left picture shows partial paths colored by 0 and 1, middle picture partial paths colored by 0 and 2, and right picture paths colored by 0 and 3. Regularly repeating patterns are marked by ellipses. Formal description is presented in the Appendix.

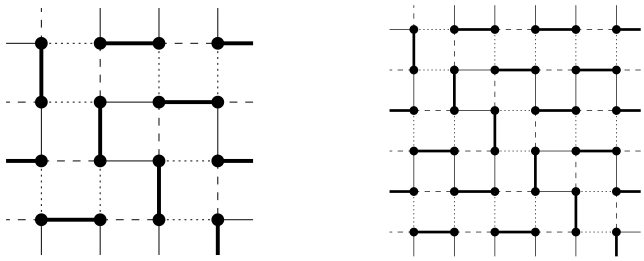

Fig. 8. Colorings of embedded tori ET_4 and ET_6 used in Lemma 3. Color 0 is printed as bold lines, color 1 as thin lines, color 2 as dotted lines and color 3 as dashed lines.

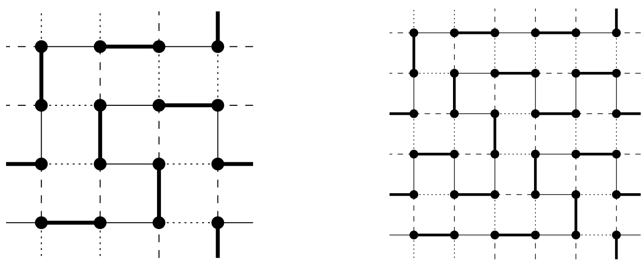

Fig. 9. Colorings of the tori T_4 and T_6 that induce a semi-perfect 1-factorization.

that this coloring is consistent with the external edges of ET' and preserves the invariant about the connectivity of the partial paths presented on Figure 7. Also, it is easy to see that each edge of ET is covered by some partial path. □

Theorem 3. *Let T_n be a torus of size $n \times n$ such that $n \geq 4$ is even. Then T_n admits a semi-perfect 1-factorization.*

Proof. We find an edge 4-coloring of the torus T_n that induces a semi-perfect 1-factorization. In case $n = 4$ or $n = 6$ the appropriate 4-coloring is shown on Figure 9.

Assume that $n \geq 8$. We obtain a coloring of a torus T of size $n \times n$ similarly as in Lemma 3. Let ET' be an embedded torus of size $n - 4 \times n - 4$ colored according to Lemma 3. We insert a torus ET' into T such that a vertex $[i, j]$ is mapped onto the vertex $[i + 2, j + 2]$ and color the remaining edges according to the Figure 10. It is easy to see that this coloring is consistent with the coloring of the external edges of ET'.

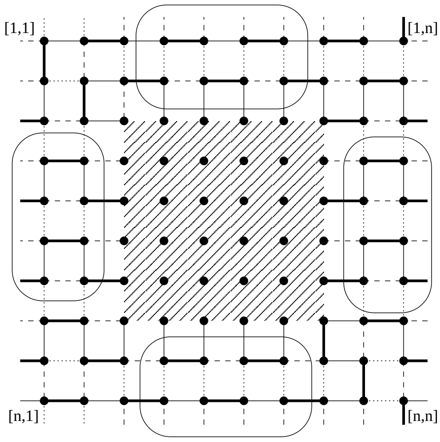

Fig. 10. Edge 4-coloring of a torus used in Theorem 3. Color 0 is printed as bold lines, color 1 as thin lines, color 2 as dotted lines and color 3 as dashed lines. Regularly repeating patterns are marked by ellipses. Formal description of this coloring is presented in the Appendix.

It remains to show that invariants about partial paths granted by Lemma 3 and the coloring presented on Figure 10 ensure that the edges of color 0 together with the edges of any other color form a Hamiltonian cycle of T_n.

Analogically to Lemma 3, it is easy to see that all non-wrap-around edges of T are covered by some partial path of the embedded torus corresponding to the torus T. Hence, to prove that edges of colors 0 and $c \in \{1, 2, 3\}$ form a Hamiltonian cycle, it is sufficient to show that all wrap-around edges of colors 0 and c are covered by one cycle consisting of edges of colors 0 and c.

For $c = 1$ it is easy to verify this fact. Indeed, the wrap-around edges of colors 0 and 1 occur on a cycle in the following order: $\{[n, n], [n, 1]\}, \{[n-1, n], [n-1, 1]\}, \{[n-3, n], [n-3, 1]\}, \ldots, \{[n-2i-1, n], [n-2i-1, 1]\}, \ldots, \{[3, n], [3, 1]\}, \{[1, n], [n, n]\}$

Similar statement holds for $c = 2$. The order of the wrap-around edges is: $\{[1, 1], [n, 1]\}, \{[n, 2], [1, 2]\}, \{[1, n], [n, n]\}, \{[n-1, n], [n-1, 1]\}, \{[n-3, n], [n-3, 1]\}, \ldots, \{[n-2i-1, n], [n-2i-1, 1]\}, \ldots, \{[3, n], [3, 1]\}$

And the order of wrap-around edges for $c = 3$ is: $\{[n-1, n], [n-1, 1]\}, \{[n-3, n], [n-3, 1]\}, \ldots, \{[n-2i-1, n], [n-2i-1, 1]\}, \ldots, \{[3, n], [3, 1]\}, \{[1, 3], [n, 3]\}, \{[1, 4], [n, 4]\}, \ldots, \{[1, i], [n, i]\}, \ldots, \{[1, n-1], [n, n-1]\}, \{[n-2, n], [n-2, 1]\}, \{[n-4, n], [n-4, 1]\}, \ldots, \{[n-2i, n], [n-2i, 1]\}, \ldots, \{[2, n], [2, 1]\}, \{[1, 1], [1, n]\}, \{[1, n], [n, n]\}$

Hence, we have proven that for every color $c \in \{1, 2, 3\}$ the edges of colors 0 or c form a Hamiltonian cycle of T_n. □

6 Conclusions and further research

We have shown that complete graphs K_{2n}, hypercubes Q_{2n+1} and tori $T_{2n\times 2n}$ admit a semi-perfect 1-factorization. The case of Q_{2n} remains open. As it seems, there is no direct relation between perfect and semi-perfect 1-factorizations, however it might be interesting to find another topologies admitting semi-perfect 1-factorization. Also, the potential of semi-perfect 1-factorizations in the topology design could be investigated.

References

1. Dan Archdeacon: Problems in Topological Graph Theory http://www.emba.uvm.edu/˜archdeac/problems/perfectq.htm
2. J. Aubert, B. Schneider: *Decompositions de la somme cartsienne d'un cycle et l'union de deux cycles hamiltoniens en cycles hamiltoniens*, Discrete Mathematics, 38(1982)7-16.
3. S. Baatz et al.: *Building Efficient Bluetooth Scatternet Topologies from 1-Factors*, Proc. IASTED International Conference on Wireless and Optical Communications, WOC 2002, pp. 300–305, Banff, Alberta, Canada, July 2002
4. Douglas W. Bass, I. Hal Sudborough: *Hamilton Decompositions and* $(n/2)$-*Factorizations of Hypercubes*, Journal of Graph Algorithms and Applications, 7(2003)79–98
5. D. Craft: *On the genus of joins and compositions of graphs*, Discrete Mathematics, 178(1998)25–50.
6. D. Craft, private communication
7. A. Kotzig: Hamilton graphs and Hamilton circuits Theory of Graphs and its Applications Proceedings of the Symposium of Smolenice, 1963 Nakl. ČSAV Praha, 1964, pp. 62–82
8. E. Mendelsohn, A. Rosa: One-Factorizations of the Complete Graph – a Survey Journal of Graph Theory, 9, 1985, 43–65

9. K. Okuda, S. W. Song: *Revisiting Hamiltonian Decomposition of the Hypercube*, Proc. XIII Symposium on Integrated Circuits and System Design, SBCCI2000, pp. 55–60, Manaus, Brazil, September 2000.
10. D.A. Pike: Hamiltonian decompositions of line graphs of perfectly 1-factorizable graphs of even degree Australian Journal of Combinatorics 12 (1995), 291–294
11. T.Tillson: A Hamiltonian decomposition of K_{2n}, $2n \geq 8$, J. Combin. Theory Ser. B 29 (1980) 68–74.

Appendix – Formal Formulation of Invariants

Formal version of Lemma 3

Let ET_n be an embedded torus of size $n \times n$, such that $n \geq 4$ is even. Let $c(e)$ be the color of the edge $e \in E(ET_n)$ and $m_c(e)$ be the last edge of the $(0, c)$-colored partial path beginning with the external edge $e \in E(ET_n)$. Then it is possible to color edges of ET_n such that all edges are covered by some partial path and the following conditions hold:

Top edges:

$$\begin{array}{lll}
c([0,1],[1,1]) & = 3 & \\
c([0,i],[1,i]) & = 1 & \forall 2 \leq i \leq n
\end{array}$$

Left edges:

$$\begin{array}{lll}
c([1,0],[1,1]) & = 1 & \\
c([2i,0],[2i,1]) & = 3 & \forall 1 \leq i \leq \frac{n}{2} \\
c([2i+1,0],[2i+1,1]) & = 0 & \forall 1 \leq i \leq \frac{n}{2} - 1
\end{array}$$

Bottom edges:

$$\begin{array}{lll}
c([n,i],[n+1,i]) & = 1 & \forall 1 \leq i \leq n-1 \\
c([n,n],[n+1,n]) & = 0 &
\end{array}$$

Right edges:

$$\begin{array}{lll}
c([2i+1,n],[2i+1,n+1]) & = 0 & \forall 0 \leq i \leq \frac{n}{2} - 1 \\
c([2i,n],[2i,n+1]) & = 3 & \forall 1 \leq i \leq \frac{n}{2} - 1 \\
c([n,n],[n,n+1]) & = 1 &
\end{array}$$

Partial paths:

$$\begin{array}{lll}
m_1([1,0],[1,1]) & = \{[3,0],[3,1]\} & \\
m_1([2i+1,0],[2i+1,1]) & = \{[2i-1,n],[2i-1,n+1]\} & \forall 2 \leq i \leq \frac{n}{2} - 1 \\
m_1([n,2i+1],[n+1,2i+1]) & = \{[n,2i+2],[n+1,2i+2]\} & \forall 0 \leq i \leq \frac{n}{2} - 2 \\
m_1([n,n-1],[n+1,n-1]) & = \{[n-1,n],[n-1,n+1]\} & \\
m_1([n,n],[n+1,n]) & = \{[n,n],[n,n+1]\} & \\
m_1([0,n],[1,n]) & = \{[1,n],[1,n+1]\} & \\
m_1([0,2i],[1,2i]) & = \{[0,2i+1],[1,2i+1]\} & \forall 1 \leq i \leq \frac{n}{2} - 1 \\
m_2([2i+1,0],[2i+1,1]) & = \{[2i-1,n],[2i-1,n+1]\} & \forall 1 \leq i \leq \frac{n}{2} - 1 \\
m_c([n,n],[n+1,n]) & = \{[n-1,n],[n-1,n+1]\} & \forall c \in \{2,3\} \\
m_3([0,1],[1,1]) & = \{[2,0],[2,1]\} & \\
m_3([i,0],[i,1]) & = \{[i-2,n],[i-2,n+1]\} & \forall 3 \leq i \leq n
\end{array}$$

Proof. The proof is equivalent to the proof written in Section 5. The following coloring of the internal edges of ET not covered by ET' is used:

Top edges:

$$
\begin{aligned}
&c([1,1],[1,2]) &&= 2 \\
&c([1,2i],[1,2i+1]) &&= 0 && \forall 1 \le i \le \tfrac{n}{2}-1 \\
&c([1,2i+1],[1,2i+2]) &&= 3 && \forall 1 \le i \le \tfrac{n}{2}-1 \\
&c([1,1],[2,1]) &&= 0 \\
&c([1,2],[2,2]) &&= 3 \\
&c([1,i],[2,i]) &&= 2 && \forall 3 \le i \le n \\
&c([2,1],[2,2]) &&= 2 \\
&c([2,2],[2,3]) &&= 1 \\
&c([2,2i+1],[2,2i+2]) &&= 0 && \forall 1 \le i \le \tfrac{n}{2}-1 \\
&c([2,2i],[2,2i+1]) &&= 3 && \forall 2 \le i \le \tfrac{n}{2}-1
\end{aligned}
$$

Bottom edges:

$$
\begin{aligned}
&c([n,2i+1],[n,2i+2]) &&= 0 && \forall 0 \le i \le \tfrac{n}{2}-2 \\
&c([n,2i],[n,2i+1]) &&= 3 && \forall 1 \le i \le \tfrac{n}{2}-1 \\
&c([n,n-1],[n,n]) &&= 2 \\
&c([n-1,i],[n,i]) &&= 2 && \forall 1 \le i \le n-2 \\
&c([n-1,n-1],[n,n-1]) &&= 0 \\
&c([n-1,n],[n,n]) &&= 3 \\
&c([n-1,2i+1],[n-1,2i+2]) &&= 3 && \forall 0 \le i \le \tfrac{n}{2}-2 \\
&c([n-1,2i],[n-1,2i+1]) &&= 0 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([n-1,n-2],[n-1,n-1]) &&= 1 \\
&c([n-1,n-1],[n-1,n]) &&= 2
\end{aligned}
$$

Left edges:

$$
\begin{aligned}
&c([2,1],[3,1]) &&= 1 \\
&c([2,2],[3,2]) &&= 0 \\
&c([2i+1,1],[2i+1,2]) &&= 3 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([2i,1],[2i,2]) &&= 0 && \forall 2 \le i \le \tfrac{n}{2}-1 \\
&c([2i+1,1],[2i+2,1]) &&= 2 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([2i+1,2],[2i+2,2]) &&= 2 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([2i,1],[2i+1,1]) &&= 1 && \forall 2 \le i \le \tfrac{n}{2}-1 \\
&c([2i,2],[2i+1,2]) &&= 1 && \forall 2 \le i \le \tfrac{n}{2}-1
\end{aligned}
$$

Right edges:

$$
\begin{aligned}
&c([n-2,n-1],[n-1,n-1]) &&= 3 \\
&c([n-2,n],[n-1,n]) &&= 1 \\
&c([2i+1,n-1],[2i+1,n]) &&= 3 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([2i,n-1],[2i,n]) &&= 0 && \forall 2 \le i \le \tfrac{n}{2}-1 \\
&c([2i+1,n-1],[2i+2,n-1]) &&= 2 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([2i+1,n],[2i+2,n]) &&= 2 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([2i,n-1],[2i+1,n-1]) &&= 1 && \forall 1 \le i \le \tfrac{n}{2}-2 \\
&c([2i,n],[2i+1,n]) &&= 1 && \forall 1 \le i \le \tfrac{n}{2}-2
\end{aligned}
$$

Formal description of Figure 10

Let T be a torus of size $n \times n$ and ET' be an embedded torus of size $(n-4)\times(n-4)$ as in the proof of Theorem 3. Let $c(e)$ be the color of the edge $e \in E(T)$. The coloring of T used in the proof of Theorem 3 presented on the Figure 10 satisfies the following conditions:

Wrap-around edges

$$
\begin{array}{lll}
c([1,1],[n,1]) & = 2 & \\
c([1,2],[n,2]) & = 2 & \\
c([1,i],[n,i]) & = 3 & \forall 3 \le i \le n-1 \\
c([1,n],[n,n]) & = 0 & \\
c([1,1],[1,n]) & = 3 & \\
c([2i,1],[2i,n]) & = 3 & \forall 1 \le i \le \frac{n}{2}-1 \\
c([2i+1,1],[2i+1,n]) & = 0 & \forall 1 \le i \le \frac{n}{2}-1 \\
c([n,1],[n,n]) & = 1 &
\end{array}
$$

Edges on the outermost perimeter

Top edges:

$$
\begin{array}{lll}
c([1,2i+1],[1,2i+2]) & = 1 & \forall 0 \le i \le \frac{n}{2}-1 \\
c([1,2i],[1,2i+1]) & = 0 & \forall 1 \le i \le \frac{n}{2}-1
\end{array}
$$

Left edges:

$$
\begin{array}{lll}
c([1,1],[2,1]) & = 0 & \\
c([2i,1],[2i+1,1]) & = 1 & \forall 1 \le i \le \frac{n}{2}-1 \\
c([2i+1,1],[2i+2,1]) & = 2 & \forall 1 \le i \le \frac{n}{2}-2 \\
c([n-1,1],[n,1]) & = 3 &
\end{array}
$$

Bottom edges:

$$
\begin{array}{lll}
c([n,2i+1],[n,2i+2]) & = 0 & \forall 0 \le i \le \frac{n}{2}-2 \\
c([n,2i],[n,2i+1]) & = 1 & \forall 1 \le i \le \frac{n}{2}-1 \\
c([n,n-1],[n,n]) & = 2 &
\end{array}
$$

Right edges:

$$
\begin{array}{lll}
c([2i+1,n],[2i+2,n]) & = 2 & \forall 0 \le i \le \frac{n}{2}-2 \\
c([2i,n],[2i+1,n]) & = 1 & \forall 1 \le i \le \frac{n}{2}-1 \\
c([n-1,n],[n,n]) & = 3 &
\end{array}
$$

Edges between the outermost and the 2nd outermost perimeter

Top edges:

$$
\begin{array}{lll}
c([1,2],[2,2]) & = 3 & \\
c([1,i],[2,i]) & = 2 & \forall 3 \le i \le n-1
\end{array}
$$

Left edges:

$$
\begin{array}{lll}
c([2,1],[2,2]) & = 2 & \\
c([2i+1,1],[2i+1,2]) & = 3 & \forall 1 \le i \le \frac{n}{2}-2 \\
c([2i,1],[2i,2]) & = 0 & \forall 2 \le i \le \frac{n}{2}-1 \\
c([n-1,1],[n-1,2]) & = 2 &
\end{array}
$$

Bottom edges:

$$\begin{aligned} &c([n-1,2],[n,2]) &&= 3 \\ &c([n-1,i],[n,i]) &&= 2 \quad \forall 3 \leq i \leq n-2 \\ &c([n-1,n-1],[n,n-1]) &&= 0 \end{aligned}$$

Right edges:

$$\begin{aligned} &c([2i,n-1],[2i,n]) &&= 0 \quad \forall 1 \leq i \leq \tfrac{n}{2}-1 \\ &c([2i+1,n-1],[2i+1,n]) &&= 3 \quad \forall 1 \leq i \leq \tfrac{n}{2}-2 \\ &c([n-1,n-1],[n-1,n]) &&= 2 \end{aligned}$$

Edges on the 2nd outermost perimeter

Top edges:

$$\begin{aligned} &c([2,2],[2,3]) &&= 1 \\ &c([2,2i+1],[2,2i+2]) &&= 0 \quad \forall 1 \leq i \leq \tfrac{n}{2}-2 \\ &c([2,2i],[2,2i+1]) &&= 3 \quad \forall 2 \leq i \leq \tfrac{n}{2}-1 \end{aligned}$$

Left edges:

$$\begin{aligned} &c([2,2],[3,2]) &&= 0 \\ &c([2i+1,2],[2i+2,2]) &&= 2 \quad \forall 1 \leq i \leq \tfrac{n}{2}-2 \\ &c([2i,2],[2i+1,2]) &&= 1 \quad \forall 2 \leq i \leq \tfrac{n}{2}-1 \end{aligned}$$

Bottom edges:

$$\begin{aligned} &c([n-1,2i],[n-1,2i+1]) &&= 0 \quad \forall 1 \leq i \leq \tfrac{n}{2}-2 \\ &c([n-1,2i+1],[n-1,2i+2]) &&= 3 \quad \forall 1 \leq i \leq \tfrac{n}{2}-2 \\ &c([n-1,n-2],[n-1,n-1]) &&= 1 \end{aligned}$$

Right edges:

$$\begin{aligned} &c([2i,n-1],[2i+1,n-1]) &&= 1 \quad \forall 1 \leq i \leq \tfrac{n}{2}-2 \\ &c([2i+1,n-1],[2i+2,n-1]) &&= 2 \quad \forall 1 \leq i \leq \tfrac{n}{2}-2 \\ &c([n-2,n-1],[n-1,n-1]) &&= 3 \end{aligned}$$